# ON THE MOTIVIC COHOMOLOGY OF $\mathbb{Z}/p^n$

TESS BOUIS

**Abstract**

We revisit certain known computations of algebraic $K$-theory in terms of the motivic complexes introduced in [EM23, Bou24], focusing on certain classes of schemes for which motivic cohomology was previously not defined.

## Contents



## 1 Perfect and semiperfect rings

Let $p$ be a prime number. It was proved by Kratzer [Kra80, Corollary 5.5] that for every perfect $\mathbb{F}_p$-algebra $R$ and every integer $n \geq 1$, the $K$-group $\mathrm{K}_n(R)$ is uniquely $p$-divisible (see also [AMM22] for a mixed characteristic generalisation). It was also proved by Kelly–Morrow that for every $\mathbb{F}_p$-algebra $R$ with perfection $R_{\mathrm{perf}}$, the natural map $\mathrm{K}(R) \to \mathrm{K}(R_{\mathrm{perf}})$ is an equivalence after inverting $p$ ([KM21, Lemma 4.1], see also [EK20, Example 2.1.11] and [Cou23, Theorem 3.1.2 and Proposition 3.3.1] for different proofs). The following result is a motivic refinement of these two facts.

**Theorem 1.1** (Motivic cohomology of perfect $\mathbb{F}_p$-schemes, after [EM23])**.** *Let $X$ be a qcqs $\mathbb{F}_p$-scheme.*

*(1) For every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X)[\tfrac{1}{p}] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}[\frac{1}{p}])$.*

*(2) For every integer $i \geq 1$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By [EM23, Theorem 4.24 (5)],[1] for every integer $i \geq 0$, the natural map

$$\phi_X^* : \mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

[1] This result is proved as a consequence of the same result in classical motivic cohomology [GL00] and in syntomic cohomology [AMMN22], and ultimately goes back to the fact that the Frobenius acts by multiplication by $p^i$ on the logarithmic de Rham–Witt sheaf $W\Omega^i_{\log}$.

induced by the absolute Frobenius $\phi_X : X \to X$ of a qcqs $\mathbb{F}_p$-scheme $X$ is multiplication by $p^i$. In particular, this natural map is an equivalence after inverting $p$, and (1) is a consequence of this and the fact that the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$ is finitary ([EM23, Theorem 4.24 (4)]). Similarly, the same result applied to the perfect $\mathbb{F}_p$-scheme $X_{\mathrm{perf}}$ implies that multiplication by $p^i$ on the complex $\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \in \mathcal{D}(\mathbb{Z})$ is an equivalence. If $i \geq 1$, this is equivalent to the fact that the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}}) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X_{\mathrm{perf}})[\tfrac{1}{p}]$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. □

**Remark 1.2** (Negative $K$-groups of perfect $\mathbb{F}_p$-algebras)**.** It is possible to construct examples of perfect $\mathbb{F}_p$-algebras whose negative $K$-groups are not $p$-divisible ([Cou23, Section 3.3]). Theorem 1.1 (2) states that the only non-$p$-divisible information in the negative $K$-groups of a perfect $\mathbb{F}_p$-algebra $R$ actually come from weight zero motivic cohomology, *i.e.*, from the complex $R\Gamma_{\mathrm{cdh}}(R, \mathbb{Z})$ ([Bou24, Example 4.68]).

Recall that a $\mathbb{F}_p$-algebra is *semiperfect* if its Frobenius is surjective.

**Corollary 1.3** (Motivic cohomology of semiperfect $\mathbb{F}_p$-algebras)**.** *Let $S$ be a semiperfect $\mathbb{F}_p$-algebra. Then for every integer $i \geq 1$, the natural commutative diagram*

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(S) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(S_{\mathrm{perf}}) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{syn}}(S_{\mathrm{perf}}) \end{array}$$

*is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* It suffices to prove the result modulo $p$, and after inverting $p$. After inverting $p$, the vertical maps become equivalences by Theorem 1.1 (1) (and the same argument for syntomic cohomology). We prove now the result modulo $p$. By Theorem 1.1 (2) (and the same argument for syntomic cohomology), the bottom terms of the commutative diagram are zero modulo $p$, so it suffices to prove that the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(S) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(S)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. By [EM23, Corollary 4.32] (see also [Bou24, Theorem 5.10] for a mixed characteristic generalisation), this is equivalent to the fact that

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \simeq 0$$

in the derived category $\mathcal{D}(\mathbb{F}_p)$. By definition, the Frobenius map $\phi_S : S \to S$ is surjective, and has nilpotent kernel. The presheaf $R\Gamma_{\mathrm{cdh}}(-, \widetilde{\nu}(i))[-i-1]$ is a finitary cdh sheaf, so the natural map

$$R\Gamma_{\mathrm{cdh}}(S, \widetilde{\nu}(i))[-i-1] \longrightarrow R\Gamma_{\mathrm{cdh}}(S_{\mathrm{perf}}, \widetilde{\nu}(i))[-i-1]$$

is then an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. The target of this map is zero by Theorem 1.1 (2) (where we use that $i \geq 1$, and the same argument for syntomic cohomology), and applying [EM23, Corollary 4.32] to the perfect $\mathbb{F}_p$-algebra $S_{\mathrm{perf}}$. □

## 2 Finite chain rings

Finite chain rings are commutative rings $\mathcal{O}_K/\pi^n$, where $\mathcal{O}_K$ is a mixed characteristic discrete valuation ring with finite residue field, $\pi$ is a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ is an integer. Examples of finite chain rings thus include finite fields, rings of the form $\mathbb{Z}/p^n$, and truncated polynomials over a finite field.

**Lemma 2.1.** *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0,p)$ and with finite residue field $\mathbb{F}_q$, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq \begin{cases} \mathbb{Z}[0] & \text{if } i = 0 \\ \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \oplus \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\frac{1}{p}] & \text{if } i \geq 1 \end{cases}$$

*in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The result for $i = 0$ follows from the equivalences

$$\mathbb{Z}(0)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \simeq R\Gamma_{\mathrm{cdh}}(\mathcal{O}_K/\pi^n, \mathbb{Z}) \simeq R\Gamma_{\mathrm{cdh}}(\mathbb{F}_q, \mathbb{Z}) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$, the first equivalence being [Bou24, Example 4.68], the second equivalence being nilpotent invariance of cdh sheaves, and the last equivalence being a consequence of the fact that fields are local for the cdh topology.

For every integer $i \geq 0$, the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathcal{O}_K/\pi^n) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) & \longrightarrow & \mathbb{Z}_p(i)^{\mathrm{BMS}}(\mathbb{F}_q) \end{array}$$

is a cartesian square in the derived category $\mathcal{D}(\mathbb{Z})$ ([Bou24, Proposition 3.29]). If $i \geq 1$, the bottom right term vanishes (use for instance the description of Bhatt–Morrow–Scholze's syntomic cohomology in characteristic $p$ in terms of logarithmic de Rham–Witt forms), and there is a natural equivalence

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{F}_q)[\tfrac{1}{p}]$$

in the derived category $\mathcal{D}(\mathbb{Z})$ (by a classical result in motivic cohomology, see also Theorem 1.1 (2) for a more general statement), hence the desired result. □

**Proposition 2.2.** *Let $\mathcal{O}_K$ be a mixed characteristic discrete valuation ring with finite residue field, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $m \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{K}_m(\mathcal{O}_K/\pi^n) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0 \\ \mathrm{H}^1_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-1,\ i \geq 1 \\ \mathrm{H}^2_{\mathrm{mot}}(\mathcal{O}_K/\pi^n, \mathbb{Z}(i)) & \text{if } m = 2i-2,\ i \geq 2 \\ 0 & \text{if } m < 0 \end{cases}$$

*of abelian groups.*

*Proof.* Let $p$ be the residue characteristic of the discrete valuation ring $\mathcal{O}_K$. The result with $p$-adic coefficients is [AKN24, Corollary 2.16]. The result with $\mathbb{Z}[\frac{1}{p}]$-coefficients reduces to the case $n = 1$, where the result follows from the description of the (classical) motivic cohomology of finite fields. The integral result is then a consequence of Lemma 2.1. □

**Theorem 2.3** (Motivic cohomology of finite chain rings, after [AKN24])**.** *Let $\mathcal{O}_K$ be a discrete valuation ring of mixed characteristic $(0,p)$ and with finite residue field $\mathbb{F}_q$, $\pi$ be a uniformizer of $\mathcal{O}_K$, and $n \geq 1$ be an integer. Then for every integer $i \geq 4p^n$,*[2] *the motivic complex*

$$\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K/\pi^n) \in \mathcal{D}(\mathbb{Z})$$

*is concentrated in degree one, where it is given by a group of order $(q^i - 1)q^{i(n-1)}$.*

[2] Note that this is not an optimal lower bound on the integer $i$. See [AKN24, Theorem 1.4] for a more precise result, in terms of the ramification index of $\mathcal{O}_K$.

*Proof.* This is a consequence of Lemma 2.1, the classical computation of the motivic cohomology of $\mathbb{F}_q$, and [AKN24, Theorem 1.4 and Proposition 1.5]. ☐

**Remark 2.4** (Nilpotence of $v_1$)**.** Antieau–Krause–Nikolaus also determine the nilpotence degree of the element $v_1$ in the mod $p$ syntomic cohomology of $\mathbb{Z}/p^n$ ([AKN24, Theorem 1.8]). This is a refinement of the key result in the study of $K(1)$-local $K$-theory of Bhatt–Clausen–Mathew [BCM20]. Note that this result on the nilpotence degree of $v_1$ can be reformulated, via Lemma 2.1, as a statement on the mod $p$ motivic cohomology of $\mathbb{Z}/p^n$.

# 3 Valuation rings

Recall that a valuation ring is an integral domain $V$ such that for any elements $f$ and $g$ in $V$, either $f \in gV$ or $g \in fV$. In recent years, valuation rings have been used as a way to bypass resolution of singularities, in order to adapt arguments from characteristic zero to more general contexts [KST21, KM21, Bou23, BEM]. In this section, we describe the motivic cohomology of valuation rings (Theorems 3.1 and 3.6). We start with the following result, stating that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, on henselian valuation rings, have a description purely in terms of algebraic cycles. See [EM23, Section 9] for related results over a field.

**Theorem 3.1.** *Let $V$ be a henselian valuation ring. Then for every integer $i \geq 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is in degrees at most $i$, and the lisse-motivic comparison map ([Bou25, Definition 2.1])*

$$\mathbb{Z}(i)^{\mathrm{lisse}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(V)$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* The second statement already appears in the proof of [Bou25, Lemma 3.25]. As in [Bou25, Lemma 3.25] or [Bou25, Corollary 2.12], the first statement is then a consequence of [Gei04, Corollary 4.4]. ☐

**Example 3.2.** Let $V$ be a henselian valuation ring. By [Bou24, Example 4.68], there is a natural equivalence

$$\mathbb{Z}(0)^{\mathrm{mot}}(V) \simeq \mathbb{Z}[0]$$

in the derived category $\mathcal{D}(\mathbb{Z})$. Similarly, Theorem 3.1, [Bou24, Example 3.9], and the fact that the Picard group of a local ring is zero, imply that the motivic complex $\mathbb{Z}(1)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ is concentrated in degree one, where it is given by

$$\mathrm{H}^1_{\mathrm{mot}}(V, \mathbb{Z}(1)) \cong V^{\times}.$$

We now apply the results of the previous sections to give an alternative description of the motivic cohomology of valuation rings with finite coefficients. The following proposition will be used to reformulate the results of [Bou23] on syntomic cohomology in terms of motivic cohomology.

**Proposition 3.3.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i}\, \mathbb{Z}/p^k(i)^{\mathrm{syn}}(V)$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* Henselian valuation rings are local rings for the cdh topology, so this is a consequence of [Bou24, Theorem 5.10]. ☐

The following result is an analogue for valuation rings of Geisser–Levine's description of motivic cohomology of smooth $\mathbb{F}_p$-algebras [GL00]. It can be deduced from the results of Kelly–Morrow [KM21] and Elmanto–Morrow [EM23].

**Theorem 3.4.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring of characteristic $p$. Then for any integers $i \geq 0$ and $k \geq 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} W_k\Omega^i_{V,\log}[-i]$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* Valuation rings of characteristic $p$ are Cartier smooth over $\mathbb{F}_p$ by results of Gabber–Ramero and Gabber ([Bou23, Theorem 3.4]), so this is a consequence of [LM23, Proposition 5.1 (*ii*)] and Proposition 3.3. □

We then prove a mixed characteristic version of Theorem 3.4, starting with the following $\ell$-adic general result.

**Proposition 3.5.** *Let $p$ be a prime number, and $V$ be a henselian valuation ring such that $p$ is invertible in $V$. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map ([Bou24, Definition 5.6]) naturally factors through an equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\text{ét}}(\mathrm{Spec}(V), \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.*

*Proof.* By Proposition 3.3, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z}/p^k)$ is in degrees at most $i$, so the result is a consequence of [Bou24, Corollary 5.6]. □

The following result generalises Proposition 3.5 when $p$ is not necessarily invertible in the valuation ring $V$, at least over a perfectoid base.

**Theorem 3.6** (Motivic cohomology of valuation rings with finite coefficients)**.** *Let $p$ be a prime number, $V_0$ be a $p$-torsionfree valuation ring whose $p$-completion is a perfectoid ring, and $V$ be a henselian valuation ring extension of $V_0$. Then for any integers $i \geq 0$ and $k \geq 1$, the Beilinson–Lichtenbaum comparison map ([Bou24, Definition 5.3]) induces a natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(V) \longrightarrow \tau^{\leq i} R\Gamma_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i})$$

*in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which is an isomorphism in degrees less than or equal to $i-1$. On $\mathrm{H}^i$, this map is injective, with image generated by symbols, via the symbol map*

$$(V^\times)^{\otimes i} \to \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}).$$

*Proof.* The fact that the Beilinson–Lichtenbaum comparison map factors through the complex

$$\tau^{\leq i} R\Gamma_{\text{ét}}(\mathrm{Spec}(V[\tfrac{1}{p}]), \mu_{p^k}^{\otimes i}) \in \mathcal{D}(\mathbb{Z}/p^k)$$

is a consequence of Proposition 3.3. The isomorphism in degrees less than or equal to $i-1$ and the injectivity in degree $i$ of this map are then a consequence of [Bou23, Theorems 3.1 and 4.12]. The last statement is a consequence of the isomorphism

$$\widehat{\mathrm{K}}^{\mathrm{M}}_i(V)/p^k \xrightarrow{\cong} \mathrm{H}^i_{\mathrm{mot}}(V, \mathbb{Z}/p^k(i))$$

of abelian groups ([Bou25, Theorem 2.21 and Corollary 2.10]). □

**Remark 3.7.** The generation by symbols appearing in Theorem 3.6 was also studied in the context of syntomic cohomology of general $p$-torsionfree $F$-smooth schemes by Bhatt–Mathew [BM23]. Note that all valuation rings are conjecturally $F$-smooth, and that the proof of Theorem 3.6 adapts more generally to any henselian $F$-smooth valuation ring.

## 4 $\mathbb{C}^\star$-ALGEBRAS

By Gelfand representation theorem, the commutative $\mathbb{C}^\star$-algebras are exactly the algebras of continuous complex-valued functions $\mathscr{C}(X;\mathbb{C})$ on a compact Hausdorff space $X$. An important theorem of Cortiñas–Thom states that commutative $\mathbb{C}^\star$-algebras are $K$-regular ([CT12, Theorem 1.5]). This result was further generalised recently by Aoki to all smooth algebras over commutative $\mathbb{C}^\star$-algebras, and over a general local field ([Aok24, Theorem 8.7]). The following result is a motivic analogue of the latter result.

**Theorem 4.1** ($\mathbb{C}^\star$-algebras are motivically regular, after [CT12, Aok24])**.** *Let $X$ be a compact Hausdorff space, $F$ be a characteristic zero local field, and $A$ be a smooth $\mathscr{C}(X;F)$-algebra. Then for any integers $i \geq 0$ and $n \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(A) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1,\dots,T_n])$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* By [Aok24, Theorem 8.7 (2)], the natural map

$$\mathrm{K}(A[T_1,\dots,T_n]) \longrightarrow \mathrm{KH}(A[T_1,\dots,T_n])$$

is an equivalence of spectra for every integer $n \geq 0$. By [Bou24, Remark 3.27 and Corollary 4.60], this implies that the vertical maps in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{mot}}(A[T_1,\dots,T_n]) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathbb{A}^1}(A) & \longrightarrow & \mathbb{Z}(i)^{\mathbb{A}^1}(A[T_1,\dots,T_n]) \end{array}$$

are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$. The bottom horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$ by definition of the presheaf $\mathbb{Z}(i)^{\mathbb{A}^1}$ ([BEM], see also [Bou24, Section 6]). So the top horizontal map is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$. $\square$

## 5 TRUNCATED POLYNOMIALS

In this section, we study the motivic cohomology of truncated polynomials, *i.e.*, the motivic cohomology of commutative rings of the form $R[x]/(x^e)$. Given a $\mathcal{D}(\mathbb{Z})$-valued functor $F(-)$, a commutative ring $R$, and an integer $e \geq 1$, we use the notation

$$F(R[x]/(x^e),(x)) := \mathrm{fib}(F(R[x]/(x^e)) \longrightarrow F(R)),$$

where the map is induced by the canonical projection $R[x]/(x^e) \to R$.

The relative $K$-theory $\mathrm{K}(k[x]/(x^e),(x))$ of truncated polynomials over a perfect field $k$ of positive characteristic was computed by Hesselholt–Madsen [HM97b, HM97a], using topological restriction homology. Their calculation was reproved by Speirs [Spe20] using Nikolaus–Scholze's approach to topological cyclic homology [NS18], and by Mathew [Mat22] and Sulyma [Sul23] using Bhatt–Morrow–Scholze's filtration on topological cyclic homology [BMS19]. This last approach was then extended to mixed characteristic by Riggenbach [Rig22]. More precisely, Riggenbach used computations in prismatic cohomology to extend the previous result to a computation of the $p$-adic relative $K$-theory $\mathrm{K}(R[x]/(x^e),(x);\mathbb{Z}_p)$ of perfectoid rings $R$, and also reproved the $p$-adic part of the known description of $\mathrm{K}(\mathbb{Z}[x]/(x^e),(x))$, originally due to Angeltveit–Gerhardt–Hesselholt [AGH09].

This recent progress would seem to indicate that $K$-theory calculations using equivariant stable homotopy may be pushed further by using cohomological techniques. Note however that the calculations in [Mat22, Sul23, Rig22] are purely $p$-adic ones, as they rely on (instances of) prismatic cohomology. In fact, all of the previous integral calculations in mixed characteristic (*i.e.*, for $R$ the ring of integers of a number field) rely on a rational result of Soulé [Sou81] and Staffeldt [Sta85], who compute the ranks of the associated relative $K$-groups using equivariant homotopy theory. In this section, we revisit and extend this rational computation, and discuss some natural motivic refinements of the previous results.

All of the above calculations use trace methods, via the Dundas–Goodwillie–McCarthy theorem. We first state the corresponding results at the level of cohomology theories.

**Lemma 5.1.** *Let $R$ be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \longrightarrow \mathbb{Z}(i)^{\mathrm{TC}}(R[x]/(x^e),(x))$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a direct consequence of [Bou24, Remark 3.21], and the fact that cdh sheaves are invariant under nilpotent extensions. □

**Corollary 5.2.** *Let $R$ be a commutative ring, $e \geq 1$ be an integer, and $p$ be a prime number. Then for every integer $i \geq 0$, the natural map*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \longrightarrow \mathbb{Z}_p(i)^{\mathrm{BMS}}(R[x]/(x^e),(x))$$

*is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}_p)$.*

*Proof.* This is a consequence of Lemma 5.1. □

**Corollary 5.3.** *Let $R$ be a commutative ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 0$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\mathbb{Q}}[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

*Proof.* This is a consequence of Lemma 5.1 and cdh descent for the presheaf $\widehat{\mathbb{L}\Omega}_{-/\mathbb{Q}}$ on commutative $\mathbb{Q}$-algebras ([EM23, Lemma 4.5]). □

**Lemma 5.4.** *For every commutative ring $R$ and integer $e \geq 1$, the object*

$$\mathbb{Z}(0)^{\mathrm{mot}}\big(R[x]/(x^e),(x)\big)$$

*is zero in the derived category $\mathcal{D}(\mathbb{Z})$.*

*Proof.* This is a consequence of the fact that the motivic complex $\mathbb{Z}(0)^{\mathrm{mot}}$ is a cdh sheaf ([Bou24, Example 4.68]). □

**Lemma 5.5.** *For any integers $e \geq 1$ and $i \geq 0$, the complex*

$$\mathbb{L}\Omega^{\leq i}_{(\mathbb{Q}[x]/(x^e),(x))/\mathbb{Q}} \in \mathcal{D}(\mathbb{Q})$$

*is concentrated in degree zero, given by a $\mathbb{Q}$-vector space of dimension $e-1$.*

*Proof.* This follows from a standard argument using the natural grading of the $\mathbb{Q}$-algebra $\mathbb{Q}[x]/(x^e)$ and the $\mathbb{Q}$-linear derivation $d : \mathbb{Q}[x]/(x^e) \to \mathbb{Q}[x]/(x^e)$ given by $d(x^j) = jx^j$; see for instance the proof of [Sta85, Proposition 5]. □

**Theorem 5.6.** *Let $R$ be a commutative ring such that the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\mathbb{Q}}$ vanishes (e.g., if $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is ind-étale over $\mathbb{Q}$),*[3] *and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq (R \otimes_{\mathbb{Z}} \mathbb{Q})^{e-1}[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Q})$.*

[3] See [MM22] for more on this condition.

*Proof.* By Corollary 5.3, there is a natural equivalence

$$\mathbb{Q}(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\,\mathbb{Q}}[-1]$$

in the derived category $\mathcal{D}(\mathbb{Q})$. By the Künneth formula for derived de Rham cohomology, and because all the positive powers of the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\,\mathbb{Q}}$ vanish, there is a natural equivalence

$$\mathbb{L}\Omega^{<i}_{(R[x]/(x^e),(x))_{\mathbb{Q}}/\,\mathbb{Q}} \simeq \mathbb{L}\Omega^{<i}_{(\mathbb{Q}[x]/(x^e),(x))/\,\mathbb{Q}} \otimes_{\mathbb{Q}} R$$

in the derived category $\mathcal{D}(\mathbb{Q})$. The result is then a consequence of Lemma 5.5. □

When $R$ is the ring of integers of a number field, the following result is due to Soulé [Sou81] when $e=2$, and to Staffeldt [Sta85] for $e \geq 2$ a general integer. Their proof uses rational homotopy theory, and ultimately reduces to a computation in cyclic homology.

**Corollary 5.7.** *Let $R$ be a commutative ring such that the cotangent complex $\mathbb{L}_{(R\otimes_{\mathbb{Z}}\mathbb{Q})/\,\mathbb{Q}}$ vanishes, and $e \geq 1$ be an integer. Then for every integer $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{K}_n(R[x]/(x^e),(x);\mathbb{Q}) \cong \begin{cases} (R\otimes_{\mathbb{Z}}\mathbb{Q})^{e-1} & \text{if } n \text{ is odd and } n \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

*of abelian groups.*

*Proof.* This is a consequence of Theorem 5.6 and [Bou24, Corollary 4.60]. □

**Remark 5.8.** Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers, and $e \geq 1$ be an integer. The orders in the torsion part of the relative $K$-theory $\mathrm{K}(\mathcal{O}_K[x]/(x^e),(x))$ were completely determined in [Rig22, Remark 1.8]. It would be interesting to use this result and Theorem 5.6 to obtain an integral description of the relative motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(\mathcal{O}_K[x]/(x^e),(x))$ for all $i \geq 0$. This would in particular reprove and generalise the result for $K=\mathbb{Q}$ of Angeltveit–Gerhardt–Hesselholt [AGH09].

We also deduce from the work of Riggenbach the following motivic interpretation of the analogous result in $K$-theory ([Rig22, Theorem 1.1]).

**Theorem 5.9** (Truncated polynomials over perfectoids, after [Rig22])**.** *Let $R$ be a perfectoid ring, and $e \geq 1$ be an integer. Then for every integer $i \geq 1$, there is a natural equivalence*

$$\mathbb{Z}_p(i)^{\mathrm{mot}}(R[x]/(x^e),(x)) \simeq \mathbb{W}_{ei}(R)/V_e\mathbb{W}_i(R)[-1]$$

*in the derived category $\mathcal{D}(\mathbb{Z}_p)$, where $\mathbb{W}(R)$ denotes the big Witt vectors of $R$, and $V$ the associated Verschiebung operator.*

*Proof.* This is a consequence of [Rig22, proof of Corollary 6.5] and Corollary 5.2. □

**Remark 5.10** (Cuspidal curves)**.** The algebraic $K$-theory of cuspidal curves (*i.e.*, curves that are defined by an equation of the form $y^a - x^b$, for $a,b \geq 2$ coprime integers) was completely determined over a perfect $\mathbb{F}_p$-algebra by Hesselholt–Nikolaus [HN20], using Nikolaus–Scholze's approach [NS18] to topological cyclic homology. This result was then generalised to mixed characteristic perfectoid rings by Riggenbach [Rig23], ultimately relying on computations in relative topological Hochschild homology. It would seem that the associated Atiyah–Hirzebruch spectral sequence should degenerate in this context, thus providing a similar computation of the motivic cohomology of cuspidal curves. An interesting question would be whether these results can be reproved, or even extended to more general base rings, using techniques from prismatic cohomology and derived de Rham cohomology.

## References


[AGH09] Vigleik Angeltveit, Teena Gerhardt, and Lars Hesselholt. On the $K$-theory of truncated polynomial algebras over the integers. *J. Topol.*, 2(2):277–294, 2009.

[AKN24] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. On the $K$-theory of $\mathbb{Z}/p^n$. https://arxiv.org/abs/2405.04329, 2024.

[AMM22] Benjamin Antieau, Akhil Mathew, and Matthew Morrow. The $K$-theory of perfectoid rings. *Doc. Math.*, 27:1923–1952, 2022.

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 171(18):3707–3806, 2022.

[Aok24] Ko Aoki. $K$-theory of rings of continuous functions. https://arxiv.org/abs/2402.05257, 2024.

[BCM20] Bhargav Bhatt, Dustin Clausen, and Akhil Mathew. Remarks on $K(1)$-local $K$-theory. *Selecta Math. (N.S.)*, 26(3):Paper No. 39, 16, 2020.

[BEM] Tom Bachmann, Elden Elmanto, and Matthew Morrow. $\mathbb{A}^1$-invariant motivic cohomology of schemes. In preparation.

[BM23] Bhargav Bhatt and Akhil Mathew. Syntomic complexes and $p$-adic étale Tate twists. *Forum Math. Pi*, 11:Paper No. e1, 26, 2023.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral $p$-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[Bou23] Tess Bouis. Cartier smoothness in prismatic cohomology. *J. Reine Angew. Math.*, 805:241–282, 2023.

[Bou24] Tess Bouis. Motivic cohomology of mixed characteristic schemes. https://arxiv.org/abs/2412.06635, 2024.

[Bou25] Tess Bouis. Weibel vanishing and the projective bundle formula for mixed characteristic motivic cohomology. https://arxiv.org/abs/2507.00000, 2025.

[Cou23] Kevin Coulembier. $K$-theory and perfection. https://arxiv.org/abs/2304.01421, 2023.

[CT12] Guillermo Cortiñas and Andreas Thom. Algebraic geometry of topological spaces I. *Acta Math.*, 209(1):83–131, 2012.

[EK20] Elden Elmanto and Adeel A. Khan. Perfection in motivic homotopy theory. *Proc. Lond. Math. Soc. (3)*, 120(1):28–38, 2020.

[EM23] Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes. https://arxiv.org/abs/2309.08463, 2023.

[Gei04] Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[GL00] Thomas Geisser and Marc Levine. The $K$-theory of fields in characteristic $p$. *Invent. Math.*, 139(3):459–493, 2000.

[HM97a] Lars Hesselholt and Ib Madsen. Cyclic polytopes and the $K$-theory of truncated polynomial algebras. *Invent. Math.*, 130(1):73–97, 1997.

[HM97b] Lars Hesselholt and Ib Madsen. On the $K$-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[HN20] Lars Hesselholt and Thomas Nikolaus. Algebraic $K$-theory of planar cuspidal curves. In *$K$-theory in algebra, analysis and topology*, volume 749 of *Contemp. Math.*, pages 139–148. Amer. Math. Soc., Providence, RI, 2020.

[KM21] Shane Kelly and Matthew Morrow. $K$-theory of valuation rings. *Compos. Math.*, 157(6):1121–1142, 2021.

[Kra80] Charles Kratzer. $\lambda$-structure en $K$-théorie algébrique. *Comment. Math. Helv.*, 55(2):233–254, 1980.

[KST21] Moritz Kerz, Florian Strunk, and Georg Tamme. Towards Vorst's conjecture in positive characteristic. *Compos. Math.*, 157(6):1143–1171, 2021.

[LM23] Morten Lüders and Matthew Morrow. Milnor $K$-theory of $p$-adic rings. *J. Reine Angew. Math.*, 796:69–116, 2023.

[Mat22] Akhil Mathew. Some recent advances in topological Hochschild homology. *Bull. Lond. Math. Soc.*, 54(1):1–44, 2022.

[MM22] Shubhodip Mondal and Alapan Mukhopadhyay. Ind-étale vs formally étale. https://arxiv.org/abs/2209.09392, 2022.

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[Rig22] Noah Riggenbach. $K$-theory of truncated polynomials. https://arxiv.org/abs/2211.11110, 2022.

[Rig23] Noah Riggenbach. $K$-theory of cuspidal curves over a perfectoid base and formal analogues. *Adv. Math.*, 433:Paper No. 109289, 40, 2023.

[Sou81] Christophe Soulé. Rational $K$-theory of the dual numbers of a ring of algebraic integers. In *Algebraic $K$-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 402–408. Springer, Berlin, 1981.

[Spe20] Martin Speirs. On the $K$-theory of truncated polynomial algebras, revisited. *Adv. Math.*, 366:107083, 18, 2020.

[Sta85] Ross E. Staffeldt. Rational algebraic $K$-theory of certain truncated polynomial rings. *Proc. Amer. Math. Soc.*, 95(2):191–198, 1985.

[Sul23] Yuri J. F. Sulyma. Floor, ceiling, slopes, and $K$-theory. *Ann. $K$-Theory*, 8(3):331–354, 2023.