## HIOB-SEMINAR SOSE25 – BERKOVICH MOTIVES

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Mondays, 12h-14h, SFB Lecture Hall

## **OVERVIEW**

In order to prove the Weil conjectures, relating the geometry of complex varieties with the number of points of algebraic varieties over finite fields, Grothendieck introduced the theory of  $\ell$ -adic étale cohomology of algebraic varieties. Inspired by the underlying conjectural philosophy of motives, Voevodsky then associated to each algebraic variety X a Z-linear derived category of motives DM(X), whose  $\ell$ -adic part is related to the  $\ell$ -adic étale cohomology of X, and whose rational part is related to the algebraic K-theory of X [Voe00]. Voevodsky's programme culminated with the proof of the Bloch–Kato conjecture, relating the étale cohomology of a field with its Milnor K-theory [Voe11, Rio14].

In parallel to this algebraic story, Tate initiated in [Tat71] the development of rigid-analytic geometry, as a non-archimedean analogue of complex analytic geometry. Variants of this theory were further developed by Raynaud [Ray74] and Huber [Hub96], and the étale cohomology of these rigid-analytic spaces found applications in several related areas, typically in the proof by Harris–Taylor of the local Langlands correspondence for  $GL_n$  [HT01]. A subsequent motivic theory for rigid-analytic varieties, similar to the algebraic theory of Voevodsky, was then initiated by Ayoub [Ayo15], and further developed by Ayoub–Gallauer–Vezzani [AGV22] and Binda–Gallauer–Vezzani [BGV23].

The theory of Berkovich spaces [Ber90], and of their étale cohomology [Ber93], provides a common framework for algebraic, complex analytic, and rigid-analytic geometries. The goal of this seminar is to study the recent paper [Sch24], where Scholze constructs a theory of étale motives for arbitrary Berkovich spaces. This theory satisfies good categorical properties, and recovers the étale version of Voevodsky's theory over a discrete field, the theory of Betti sheaves over  $\mathbb{C}$ , and is closely related to Ayoub's theory over a non-archimedean field. To motivate the constructions of this paper, we will cover some of the relevant background material on analytic geometry (see for instance [Con07]) and on motivic categories (see for instance [MVW06]).

## TALKS

Talk 1: Introduction (Tess Bouis, 28.04.2025). Overview of the seminar and distribution of the talks. Please attend if you are considering giving a talk.

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Talk 2: Banach rings (Antoine Sedillot, 05.05.2025). Mainly follow [Sch24, §2]. The goal of this talk is to give intuitions about the basics of Berkovich geometry. Define the notions of a Banach ring [Sch24, Definitions 2.1 and 2.2], of the Berkovich spectrum of a Banach ring [Sch24, Definition 2.13]. Discuss (and in particular compare the algebraic and analytic variants in) the examples of Banach fields, of the integers  $\mathbb{Z}$ , of the free Banach algebras over a Banach ring, of the ball, of  $\mathbb{A}^1$ , and of  $\mathbb{P}^1$ . Several introductury texts are available online; see for instance [Bak08, §1] for the last example.

Talk 3: The arc-topology (Jeroen Hekking, 12.05.2025). Begin with a quick reminder on Grothendieck topologies, following for instance [Kha23, §2]. Briefly introduce Bhatt–Mathew's definition of the (algebraic) arc-topology [BM21, Definition 1.1] and explain the motivation for this topology via the example of étale cohomology [BM21, Theorem 1.8]. Then introduce Scholze's (analytic) arc-topology in the setting of Banach rings [Sch24, Definition 3.1] and explain quickly why this defines a Grothendieck topology on (the opposite of) the category of Banach rings [Sch24, Propositions 3.2 and 3.3]. Explain carefully the fact that strictly totally disconnected Banach rings form a basis for the arc-topology [Sch24, Definition 3.10 to Theorem 3.13], and how the situation specializes when working over  $\mathbb{C}$  [Sch24, Example 3.8 and Proposition 3.9].

Talk 4: Voevodsky/Ayoub's étale motives (Marc Hoyois, 19.05.2025). This talk is here to motivate what will happen in the next talks. Introduce Voevodsky/Ayoub's category of étale motives associated to a scheme X, and discuss its basic properties, following [Cis21, §1] or any other resource. Also state the cancellation theorem.

Talk 5: Definition of Berkovich motives (Johannes Glossner, 26.05.2025). Define small arc-stacks, and compare this to the definition of Berkovich spaces, as introduced for instance in [LP24]. Introduce the category of Berkovich motives  $\mathcal{D}_{mot}(X)$  associated to a small arc-stack X [Sch24, Definitions 4.10, 5.1, 5.2, 5.18, and 9.1]. State the cancellation theorem [Sch24, Theorem 1.9], and comment on how this allows to understand more concretely the category  $\mathcal{D}_{mot}(X)$ . Discuss some of the properties of finitary arc-sheaves from [Sch24, §4].

Talk 6: Ball-invariant arc-sheaves (Sebastian Wolf, 02.06.2025). Discuss some of the properties of ball-invariant finitary arc-sheaves from [Sch24, §5]. Compare the notions of  $\mathbb{A}^1$ -localisation and  $\mathbb{B}$ -localisation (see for instance [Sch24, Remark 5.7] and [KST19, §1.2]). Then discuss some of the categorical properties of Berkovich motives following [Sch24, §9].

Talk 7: Free motivic sheaves (Denis-Charles Cisinski, 16.06.2025). Introduce and discuss the properties of the free motivic sheaves, following [Sch24, §5.1 and §6]. Discuss in particular the comparison of Berkovich motives with finite coefficients with  $\ell$ -adic étale sheaves [Sch24, Theorem 6.7].

Talk 8: The cancellation theorem (Han-Ung Kufner, 23.06.2025). State and prove the cancellation theorem [Sch24, Theorem 7.1]. Compare to the statements and proofs in the algebraic and rigid-analytic settings (see [Voe10, Bac21] and [Ayo15, Vez17], respectively).

Talk 9: The arc-local K-theory (Marco Volpe, 30.06.2025). Recall the definition of connective K-theory of commutative rings. Say a word about how algebraic K-theory appears in Voevodsky's theory of motives. Define the arc-local K-theory  $\overline{K}(R)$  of a Banach ring R [Sch24, Definition 8.3 and Proposition 8.4], and explain the rational Adams decomposition [Sch24, Corollary 8.6]. Finally, state the second part of [Sch24, Theorem 8.13], and explain as much as possible from its proof.

Talk 10: Rigid categories (Giovanni Rossanigo, 07.07.2025). Give an introduction to the notions of compactly generated category and of rigid category, as discussed in [KNP24, §4.3 and §4.4]. Talk in particular about the categorical Künneth formula [Sch24, Corollary 10.6], state the theorem that the category of Berkovich motives  $\mathcal{D}_{mot}(X)$  is rigid under mild assumptions on X [Sch24, Proposition 10.3], and discuss its proof.

Talk 11: Comparison with Voevodsky/Ayoub's étale motives (Niklas Kipp, 14.07.2025). Prove that over a discrete field k, Berkovich motives are equivalent to Voevodsky/Ayoub's étale motivic sheaves [Sch24, Theorem 11.1]. Then explain the description of  $\mathcal{D}_{mot}(C)$ , for C the completed algebraic closure of  $k((T))_{1/2}$ , in terms of motivic nearby cycles [Sch24, Corollary 11.10], and how this could be used to give a similar description of  $\mathcal{D}_{mot}(\mathbb{C}_p)$  using [Sch24, Proposition 6.8] (see also [BGV23]).

## References

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