BEILINSON–LICHTENBAUM PHENOMENON FOR MOTIVIC COHOMOLOGY

TESS BOUIS and ARNAB KUNDU

Abstract

The goal of this paper is to study non-$\mathbb{A}^1$-invariant motivic cohomology, recently defined by Elmanto, Morrow, and the first-named author, for smooth schemes over possibly non-discrete valuation rings. We establish that the cycle class map from p-adic motivic cohomology to a suitable truncation of Bhatt–Lurie's syntomic cohomology is an isomorphism, thereby verifying the Beilinson–Lichtenbaum conjecture in this generality. As a consequence, we prove that this motivic cohomology integrally recovers the classical definition of motivic cohomology in terms of Bloch's cycle complexes, whenever the latter is defined. Over perfectoid rings, we show that this cohomology theory is actually $\mathbb{A}^1$-invariant, thus partially answering a question of Antieau–Mathew–Morrow. The key ingredient in our approach is a version of Gabber's presentation lemma applicable in mixed characteristic, non-noetherian settings.

Contents

1 Introduction

Beilinson and Lichtenbaum, inspired by Grothendieck's vision of motives, conjectured the existence of a theory of *motivic cohomology* satisfying a list of properties, including the expression of this cohomology theory as the graded pieces of a filtration on algebraic K-theory [Lic73, Lic84, Beĭ86, Beĭ87, BMS87].

Using the framework of motivic $\mathbb{A}^1$-homotopy theory, which ultimately led to the proof of the Bloch–Kato conjecture [Voe11], Voevodsky initiated the development of this cohomology theory for smooth schemes over a field [Blo86, VSF00]. This theory was later generalised to smooth schemes over any Dedekind domain by a number of authors [Lev01, Gei04, Spi18, Bac22]. In both situations, the authors showed that *Bloch's cycle complexes* $z^i(X, \bullet - 2i)$ satisfy most of Beilinson and Lichtenbaum's conjectures.

Recently, Elmanto and Morrow [EM23], using trace methods in algebraic K-theory, introduced a new definition of motivic cohomology that applies to arbitrary quasi-compact quasi-separated (qcqs) schemes over a field. Building on their work, the first-named author of the present article extended this approach to define motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ for arbitrary qcqs schemes X over $\mathbb{Z}$ [Bou24]. This theory, which satisfies various expected structural properties of motivic cohomology, may thus provide a more flexible framework for studying algebraic cycles. To make this expectation precise, it is therefore natural to ask how this theory compares to the more classical theory of Bloch's cycle complexes in the smooth case.

Our first result, given below, is to show that the complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ coincide with Bloch's cycle complexes in the smooth case, *i.e.*, that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ extend the classical definition of motivic cohomology beyond the case of smooth schemes. In particular, this suggests that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ are well-suited to understand the interplay between algebraic cycles in characteristic zero and in positive characteristic.

Theorem A (See Theorem 6.1)**.** *Let X be an ind-smooth scheme over a Dedekind domain. Then for every integer $i \geqslant 0$, the natural comparison map*

$$z^i(X, \bullet - 2i) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

For smooth schemes over a field, this result was proved by Elmanto–Morrow [EM23], using Gabber's presentation lemma [CTHK97]. Our proof of Theorem A uses a Nisnevich-local variant of Gabber's presentation lemma in mixed characteristic (Theorem E below) to reduce the statement to the case where X itself is the spectrum of a discrete valuation ring. In this case, the equivalence follows from a special case of the Beilinson–Lichtenbaum conjecture proved by Geisser [Gei04] as a consequence of the Bloch–Kato conjecture, together with a special case of Theorem B established in [Bou24], by comparing both sides to the syntomic cohomology of X.

Our next results concern possibly non-noetherian rings. These rings—most prominently perfectoid rings [Sch12, BM21, BS22]—have become a powerful tool in modern p-adic geometry. In many contemporary approaches, results for noetherian objects are indeed proved by first reducing claims to suitable non-noetherian ones. This shift provides the main motivation for studying algebraic geometry over schemes that are not necessarily noetherian.

We fix a prime number p for the rest of this introduction. We also let L_{Nis} denote the Nisnevich sheafification functor, and $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ denote the syntomic cohomology complexes, as defined by Bhatt and Lurie [BL22]. By construction, these syntomic complexes provide a generalisation of the p-adic étale cohomology complexes when p is not necessarily invertible in the input.

The following result generalises (and relies upon) the Beilinson–Lichtenbaum comparison, stating that motivic cohomology can be expressed as a suitable truncation of syntomic cohomology, from the classical setting where the base is a field or a discrete valuation ring to that of an arbitrary valuation ring.[1] Note that this comparison is not expected to hold over an arbitrary base ring (see Remark 5.2).

Theorem B (Non-noetherian Beilinson–Lichtenbaum conjecture; see Theorem 5.1)**.** *Let X be an ind-smooth scheme over a valuation ring. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, there is a natural equivalence*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \simeq \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

As a consequence of Theorem A, for smooth schemes over discrete valuation rings, Theorem B agrees with the classical statement of the Beilinson–Lichtenbaum conjecture, proved in [Gei04, Voe11].

The proof of the Beilinson–Lichtenbaum comparison for smooth schemes over fields in [Voe11] relies (among many things) on Gersten resolutions in order to reduce the statement to the case of fields (see [GL01]). Our proof of Theorem B relies on a Nisnevich-local variant of Gabber's presentation lemma which holds over an arbitrary valuation ring (Theorem E below), which we use to reduce the statement to the case of henselian valuation rings. In this case, the result is a consequence of [Bou24, Proposition 11.10].

As a consequence of Theorem B, we obtain the following description of motivic cohomology of smooth schemes over valuation rings in terms of the p-adic étale cohomology of their generic fibre.

To formulate this result, we use the notion of a F-smooth ring due to Bhatt and Mathew [BM23], which is a non-noetherian generalisation of the notion of a regular ring. Examples of F-smooth rings include regular rings (in particular, discrete valuation rings), perfectoid rings, and any valuation ring containing a field or whose p-completion contains a perfectoid valuation ring [BM23, KM21, Bou23]. Conjecturally, all valuation rings should be F-smooth (see Remark 6.6).

[1] A valuation ring is an integral domain V such that for any elements f and g in V, either $f \in gV$ or $g \in fV$. Note that a valuation ring is noetherian if and only if it is a discrete valuation ring.

Theorem C. *Let V be a p-torsionfree F-smooth valuation ring (e.g., the ring of integers of an algebraic extension of a p-adic local field), and R be a henselian local ind-smooth algebra over V. Then, for all integers $i, n \geqslant 0$ and $k \geqslant 1$, there is a natural isomorphism*

$$\mathrm{H}^n_{\mathrm{mot}}(\mathrm{Spec}(R), \mathbb{Z}/p^k(i)) \cong \begin{cases} \mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i}) & \text{if } n < i \\ \mathrm{Im}((R^\times)^{\otimes i} \longrightarrow \mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(R[\frac{1}{p}]), \mu_{p^k}^{\otimes i})) & \text{if } n = i \\ 0 & \text{if } n > i \end{cases}$$

of abelian groups.

Our proof of Theorem C proceeds in two steps: first, we use Theorem B to compare motivic cohomology with a truncation of syntomic cohomology; we then apply results of Bhatt and Mathew [BM23] in integral p-adic Hodge theory to compute this truncated syntomic cohomology in terms of étale cohomology.

As a consequence of Theorem C, we obtain the following corollary, which establishes a form of purity for motivic cohomology over a perfectoid base. We note that, in particular, our proof of Corollary D does not rely on absolute purity (see Remark 5.10).

Corollary D (See Corollary 5.9)**.** *Let V be a perfectoid valuation ring of residue characteristic p, F be the fraction field of V, and X be an ind-smooth scheme over V. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X_F)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

The case of the above result where $V = \mathcal{O}_{\overline{K}}$, for K a p-adic local field, is a cohomological refinement of a K-theoretic result proved by Nizioł to prove Fontaine's crystalline conjecture [Niz98]. This cohomological result can in turn be used to give a more streamlined version of this proof (see Remark 6.3).

As a final application of Theorem B, we demonstrate that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ are typically $\mathbb{A}^1$-invariant on smooth schemes over valuation rings (see Theorem 6.9), using recent results of Bachmann–Elmanto–Morrow [BEM25]. This partially answers a question of Antieau–Mathew–Morrow [AMM22].

We end this introduction with the following version of Gabber's presentation lemma over a general base, which serves as a technical basis for our Theorems A and B.

Theorem E (Presentation lemma; see Corollary 2.25)**.** *Let S be the spectrum of a henselian local ring, X be a smooth S-scheme, and $Z \hookrightarrow X$ be a closed immersion of S-schemes which is S-fibrewise of positive codimension. Then for every point $x \in X$, there exist a Nisnevich neighbourhood $x \in X' \to X$, a smooth S-scheme T, and a Nisnevich square of T-schemes*

$$\begin{array}{ccc} X' \setminus Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \psi \\ \mathbb{A}^1_T \setminus \psi(Z') & \hookrightarrow & \mathbb{A}^1_T \end{array}$$

where $Z' := Z \times_X X'$, ψ is étale, and such that $\psi(Z')$ is finite over T.

Note that Gabber's presentation lemma over an infinite field [Gab94] has already been extended to arbitrary fields by Hogadi–Kulkarni [HK20], to Dedekind domains with infinite residue fields by Schmidt–Strunk [SS18], and to noetherian domains with infinite residue fields by Deshmukh–Hogadi–Kulkarni–Yadav [DHKY21]. A weaker form of Theorem E, where the finiteness of $\psi(Z')$ over T is not guaranteed, was also obtained by Druzhinin [Dru22, Section 4.2].

Our proof of Theorem E extends and often simplifies the proofs of these presentation lemmas. In particular, our arguments work over a base which is not necessarily noetherian, and with no hypothesis on its residue fields. Along the way, we also study the existence of good compactifications over non-noetherian bases, which may be of independent interest (see Section 2).

Notation.

We will denote by Set_* the 1-category of pointed sets, by $[*]$ the trivial class of a pointed set, and by Sp the stable ∞-category of spectra. We always index complexes using the cohomological grading.

A presheaf is called *finitary* if it commutes with filtered colimits of rings. For the Grothendieck topologies τ that we will use, we denote by L_τ the associated sheafification functors. Given a scheme X and a $\mathcal{D}(\mathbb{Z})$-valued finitary τ-sheaf F on the small τ-site of X, we say that F lives τ-locally on X in degrees at most i if F takes values in $\mathcal{D}^{\leqslant i}(\mathbb{Z})$ on the τ-local rings of X. Given a point x in a scheme X, by a *Zariski neighbourhood* (resp. a *Nisnevich neighbourhood*) of $x \in X$, we mean an open subset $x \in X' \subseteq X$ (resp. an étale morphism $X' \to X$ together with a lift of the point x in X'). In either case, when X' is moreover affine, we call this neighbourhood an affine neighbourhood.

Unless otherwise specified, by dimension of a scheme, we mean its Krull dimension. A morphism of schemes $f\colon X \to S$ is said to be *fibrewise of dimension* d (resp. *fibrewise of dimension* $\leqslant d$), for some integer $d \geqslant 0$, if every fibre of f is of dimension d (resp. of dimension $\leqslant d$). The dimension of f is defined to be the dimension of its fibre with the maximum dimension.

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2 Nisnevich-local presentation lemma with finite residue fields

Gabber’s presentation lemma, which can be considered as an analogue in algebraic geometry of the tubular neighbourhood theorem from differential geometry, describes the geometry of smooth schemes over a field. In this section, our goal is to prove a Nisnevich-local variant of this lemma for smooth schemes over more general base schemes (Corollary 2.25). This result is the main geometric ingredient needed in Section 3 to prove Gersten’s injectivity (Theorem 3.5). To prove our presentation lemma, we follow (and improve on) the techniques of Schmidt and Strunk [SS18], who build up on results of Kai [Kai21], to establish a Nisnevich-local variant of Gabber’s presentation lemma over discrete valuation rings with infinite residue fields. Thanks to the strong geometric input from [GLL15] and [Čes22a], our proof of Corollary 2.25 is simpler than that of [SS18], and works over any commutative ring.

As in [SS18], we divide the proof into two independent steps.

(a) First, we prove the existence of good compactifications for closed immersions (in the sense of Definition 2.3 (2)) over a base ring which is stably coherent and has a stably noetherian topological space (Corollary 2.21). Note that these conditions are satisfied by noetherian rings and by valuation rings of finite rank (Examples 2.8). Previously, this was known over discrete valuation rings with infinite residue fields ([Kai21, Theorem A.1]), for local complete intersections over noetherian domains with infinite residue fields ([DHKY21]), and for local complete intersections over more general noetherian rings ([DHKY23]). Inspired by the techniques of [Čes22a], we remove the restriction on the residue fields by considering hypersurface sections instead of hyperplane sections. Moreover, using the results from [GLL15] allows us to go beyond the noetherian setting.

(b) Second, assuming the existence of such a good compactification, and using the presentation lemma over stably coherent rings constructed in [Kun25, Theorem 2.15], we produce our main geometric presentation result (Theorem 2.24).

Note that the main difference between our result and Gabber's presentation lemma over a field is that our presentation lemma is Nisnevich-local rather than Zariski-local. All the sheaves considered in this article being in fact Nisnevich sheaves, this difference will not be important for our purposes. Upgrading the results of this section to Zariski-local statements could be interesting for other applications, but it would seem to require new ideas.

We first recall the notion of Prüfer rings, which are a non-noetherian generalisation of Dedekind rings.

Definition 2.1 (Prüfer ring, [Gil92, Chapter IV, Section 22])**.** A commutative ring is said to be a *Prüfer domain* if it is an integral domain whose localisation at every prime ideal is a valuation ring. A commutative ring is said to be a *Prüfer ring* if it is a finite product of Prüfer domains.

In particular, a commutative ring is a valuation ring if and only if it is a local Prüfer ring, and is a Dedekind ring if and only if it is a noetherian Prüfer ring. The ring of algebraic integers $\overline{\mathbb{Z}}$ is an example of non-noetherian Prüfer domain of Krull dimension one, which is, in particular, not a Dedekind ring. Also note that, unlike Dedekind rings, Prüfer rings can be of arbitrary Krull dimension. For instance, the subring $\{P(X) \in \mathbb{Q}[X] \mid P(0) \in \mathbb{Z}\}$ of $\mathbb{Q}[X]$ is a Prüfer ring of Krull dimension two ([CC16, Theorem 17]), and the ring of entire holomorphic functions on the complex plane is a Prüfer ring of infinite Krull dimension ([Lop98]).

Remark 2.2. The following properties make algebraic geometry over Prüfer rings more tractable than over an arbitrary (even noetherian) base.

(1) By a result of Artin ([Gro66, lemme 14.3.10], see also [Kun23, Lemma 3.6]), a surjective irreducible scheme over a Prüfer domain is fibrewise of constant dimension.

(2) An integral domain is a Prüfer domain if and only if each of its nonzero finitely generated ideals is invertible ([Gil92, Theorem 22.1]). By [Bou72, Chapter I, 2.4, Prop. 3(ii)], this implies that a module over a Prüfer ring is flat if and only if it is torsionfree.

(3) The class of Prüfer domains is closed under filtered colimits ([Gil92, Proposition 22.6]) and under localisations ([Gil92, Proposition 22.5]).

We now introduce the notion of good compactifications that will be used in Corollary 2.21.

Definition 2.3 (Good compactification)**.** Let S be a scheme.

(1) A dense open immersion $X \hookrightarrow \overline{X}$ of S-schemes is said to be a *compactification* if the S-scheme $\overline{X}$ is projective, and is said to be a *good compactification* if for every point $s \in S$ such that the fibre $\overline{X}_s$ is non-empty, the open immersion $X_s \hookrightarrow \overline{X}_s$ satisfies $\dim(\overline{X}_s \setminus X_s) < \dim(\overline{X}_s)$, and in particular $\dim(X_s) = \dim(\overline{X}_s)$.

(2) An S-scheme X is said to admit a *good compactification* if there exists a good compactification $X \hookrightarrow \overline{X}$ of S-schemes.

(3) A closed immersion $Z \hookrightarrow X$ of S-schemes is said to admit a *good compactification* if there exists a compactification $X \hookrightarrow \overline{X}$ of S-schemes such that the schematic closure morphism $Z \hookrightarrow \overline{Z}$ inside $\overline{X}$ is a good compactification.

Definition 2.3 is inspired by the statement of [DHKY21, Theorem 2.1], where the authors show that the above notion is sufficient for the proof of the presentation lemma.

Remark 2.4. Let $X \hookrightarrow \overline{X}$ be a compactification of qcqs[2] S-schemes in the sense of Definition 2.3 (1).

(1) For every point $s \in S$, we know that $\dim(X_s) \leqslant \dim(\overline{X}_s)$. In particular, a generic point $\eta \in \overline{X}_s$ of an irreducible component of $\overline{X}_s$ of dimension $\dim(\overline{X}_s)$ that lies in X_s is automatically a generic point of X_s. To prove that $\dim(\overline{X}_s \setminus X_s) < \dim(\overline{X}_s)$, it then suffices to prove that each generic point $\eta \in \overline{X}_s$ of maximum dimension lies in X_s.

[2] By [Gro66, remarque 11.10.3 (*iv*)], an open immersion of qcqs schemes is schematically dense if and only if it is dense (*i.e.*, if the morphism of underlying topological spaces is dense).

(2) The condition given in Definition 2.3 (1) is weaker than assuming the S-fibrewise density of the compactification $X \hookrightarrow \overline{X}$. Indeed, it might happen that $\overline{X}_s$, for some $s \in S$, contains an irreducible component of dimension less than $\dim(\overline{X}_s)$ whose generic point does not lie in X_s. However, given a point $s \in S$, if the fibres X_s and $\overline{X}_s$ are both of pure dimension d, then the open immersion $X_s \hookrightarrow \overline{X}_s$ satisfies $\dim(\overline{X}_s \setminus X_s) < \dim(\overline{X}_s)$ if and only if it is dense.

(3) Given a flat S-scheme S', if $X \hookrightarrow \overline{X}$ is a compactification (resp. a good compactification) of S-schemes, then the base change morphism $X \times_S S' \hookrightarrow \overline{X} \times_S S'$ is a compactification (resp. a good compactification) of S'-schemes. In view of the preceding remarks, this is a consequence of the fact that schematic density is preserved by flat base change ([Sta25, Tag 01RE]). If the S-scheme S' is moreover projective, this implies that the good compactification of S'-schemes $X \times_S S' \hookrightarrow \overline{X} \times_S S'$ is also a good compactification of S-schemes.

(4) In Definition 2.3 (3), note that the compactification $X \hookrightarrow \overline{X}$ is not required to be a good compactification. This induces a slight relaxation on the usual hypotheses needed for the presentation lemma (Corollary 2.25).

We now prove that, Nisnevich-locally on the base scheme S, a closed immersion of finitely presented S-schemes $Z \hookrightarrow X$ admits a good compactification in the sense of Definition 2.3 (3) (Corollary 2.21). We prove this by induction on the relative dimension of Z over S. We will need the following auxiliary results.

Lemma 2.5. *Let S be a scheme, $d \geqslant 0$ be an integer, and $\iota_Y : Y \hookrightarrow \overline{Y}$ be a good compactification of S-schemes such that the scheme $\overline{Y}$ is qcqs. If the S-scheme Y is S-fibrewise of dimension d, then so is the S-scheme $\overline{Y}$. Moreover, if*

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \overline{X} \\ \downarrow f & & \downarrow \overline{f} \\ Y & \xrightarrow{\iota_Y} & \overline{Y}, \end{array} \tag{2.5.1}$$

is a cartesian square of S-schemes such that $\overline{f}$ is finite and X and Y are S-fibrewise of dimension d, then the dense open immersion $\iota_X : X \hookrightarrow \overline{X}$ is a good compactification of S-schemes.

Proof. Let s be a point of S. The first claim is a consequence of Definition 2.3 (1), and in particular of the fact that $\dim(Y_s) = \dim(\overline{Y}_s)$. We now show the second claim. A finite scheme over a projective S-scheme is a projective S-scheme ([Sta25, Tags 0B3I and 0C4P]), so the dense open immersion $\iota_X : X \hookrightarrow \overline{X}$ is a compactification of S-schemes. To prove that it is good, it suffices by Remark 2.4 (1) to prove that any generic point $\eta_X \in \overline{X}_s$ of an irreducible component of dimension $\dim(\overline{X}_s)$ lies in X_s. Finiteness is stable under base change ([Sta25, Tags 0B3I]), so the morphism $\overline{f}_s : \overline{X}_s \to \overline{Y}_s$ is finite. This implies, by the dimension formula [Sta25, Tag 02JU], that $\dim(\overline{X}_s) = \dim(\overline{Y}_s)$, and that $\eta_Y := \overline{f}_s(\eta_X)$ is a generic point of an irreducible component of $\overline{Y}_s$ of dimension $\dim(\overline{Y}_s)$. The morphism $\iota_Y : Y \hookrightarrow \overline{Y}$ being a good compactification of S-schemes, the point $\eta_Y \in \overline{Y}_s$ necessarily lies in Y_s. The fact that the point $\eta_X \in \overline{X}_s$ lies in X_s is then a consequence of the fact that square (2.5.1) is cartesian. □

In the non-noetherian setting, finite type objects are not necessarily finitely presented. The following notions will be used in Lemma 2.9 to reduce the discrepancy between objects of finite type and of finite presentation.

Definition 2.6. (1) (Coherence) A scheme X is said to be *coherent* if it is qcqs and if every affine open of X is the spectrum of a coherent ring, *i.e.*, a commutative ring in which every finitely generated ideal is finitely presented.[3] It is said to be *stably coherent* if every finitely presented X-scheme is coherent.

(2) (Noetherian topological space) A scheme X is said to have a *noetherian topological space* if its underlying topological space is noetherian. It is said to have a *stably noetherian topological space* if any finite type X-scheme has a noetherian topological space.

[3] Similarly, an $\mathcal{O}_X$-module $\mathscr{F}$ is said to be *coherent* if it is of finite type and if for every open $U \subseteq X$ and every finite collection $s_i \in \mathscr{F}(U)$, $i = 1, \ldots, n$, the kernel of the associated map $\bigoplus_{i=1,\ldots,n} \mathcal{O}_U \to \mathscr{F}$ is of finite type ([Sta25, Tag 01BV]).

Note that a scheme X with a noetherian topological space is automatically qcqs. Indeed, any open subspace of a noetherian topological space is noetherian ([Sta25, Tag 0052]), hence quasi-compact.

Remark 2.7. (1) In Definition 2.6, to verify that a scheme X has a stably noetherian topological space, it suffices to check that, for every integer $n \geq 0$, the scheme $\mathbb{A}^n_X$ has a noetherian topological space. Indeed, any X-scheme of finite type has a finite covering by affine opens, each of which can be embedded inside an affine space over X.

(2) Given a scheme X with a noetherian topological space and a closed subscheme $Z_0 \subseteq X$, there always exists a finitely presented closed subscheme $Z_1 \subseteq X$ which contains Z_0, and such that Z_0 and Z_1 have the same underlying topological space. Considering the family of finitely generated subideals of the ideal of definition of $Z_0 \subseteq X$, this follows from the definition of a noetherian topological space.

Examples 2.8. (1) A noetherian scheme is stably coherent and has a stably noetherian topological space.

(2) The spectrum of a Prüfer ring is stably coherent ([Kun23, Section 4.1], see also [Kun25, Lemma 2.12]). The spectrum of a semilocal Prüfer ring of finite Krull dimension has a stably noetherian topological space. More generally, a scheme with a finite underlying topological space has a stably noetherian topological space ([Sta25, Tag 0053]).

(3) The perfection X_{perf} of a noetherian $\mathbb{F}_p$-scheme X (or more generally of a $\mathbb{F}_p$-scheme X with stably noetherian topological space) has a stably noetherian topological space. This is a consequence of the fact that the morphism $X_{\mathrm{perf}} \to X$ is a homeomorphism on the underlying topological spaces, and of Remark 2.7 (1).

Lemma 2.9. *Let S be a scheme, $f \colon X \to Y$ be a finite morphism of finitely presented S-schemes, and $\iota_Y \colon Y \hookrightarrow \overline{Y}$ be a compactification of finitely presented S-schemes. If $\overline{Y}$ is coherent and has a stably noetherian topological space, then there exists a cartesian square of S-schemes*

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \overline{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xrightarrow{\iota_Y} & \overline{Y} \end{array}$$

such that $\overline{f}$ is finite and finitely presented.

Proof. The Y-scheme X is finite, hence projective ([Sta25, Tag 0B3I]), so there exist an integer $N \geqslant 0$ and a closed immersion $X \hookrightarrow \mathbb{P}^N_Y$. Let $\overline{X}_0$ be the schematic closure of X inside $\mathbb{P}^N_{\overline{Y}}$. Although the scheme $\overline{X}_0$ is not necessarily finitely presented over $\overline{Y}$, the scheme $\mathbb{P}^N_{\overline{Y}}$ has a noetherian topological space, and so, by Remark 2.7 (2), there exists a finitely presented closed subscheme $\overline{X}_1 \subseteq \mathbb{P}^N_{\overline{Y}}$ which contains $\overline{X}_0$, and such that $\overline{X}_0$ and $\overline{X}_1$ have the same underlying topological space. In particular, the induced morphism $\overline{f}_1 \colon \overline{X}_1 \to \overline{Y}$ is a finitely presented projective morphism of qcqs schemes. The commutative diagram of S-schemes

$$\begin{array}{ccc} X & \hookrightarrow & \overline{X}_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}_1} \\ Y & \xrightarrow{\iota_Y} & \overline{Y} \end{array}$$

is then cartesian: indeed, ι_Y is an open immersion, so this can be checked on the underlying topological spaces. Let $\overline{X}$ be the scheme $\mathrm{Spec}_{\overline{Y}}(\overline{f}_{1,*}\mathcal{O}_{\overline{X}_1})$, where $\mathrm{Spec}_{\overline{Y}}$ denotes the relative Spec over $\overline{Y}$. By [Kun25, Lemma 2.13 (*iii*)], where we use that $\overline{Y}$ is stably coherent and that $\overline{f}_1$ is finitely presented, the $\mathcal{O}_{\overline{Y}}$-module $\overline{f}_{1,*}\mathcal{O}_{\overline{X}_1}$ is coherent. In particular, the induced morphism $\overline{f} \colon \overline{X} \to \overline{Y}$ is finite. The desired cartesian square is then a consequence of the previous cartesian square and of the affineness of the morphism $\overline{f}_1$ ([Sta25, Tag 02KG]). ☐

Corollary 2.10. *Let S be a scheme, $d \geqslant 0$ be an integer, $f\colon X \to Y$ be a finite morphism of finitely presented S-schemes which are fibrewise of dimension d, and $\iota_Y\colon Y \hookrightarrow \overline{Y}$ be a good compactification of finitely presented S-schemes. If $\overline{Y}$ is coherent and has a stably noetherian topological space (e.g., if S is stably coherent and has a stably noetherian topological space), then there exists a cartesian square of S-schemes*

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \overline{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ Y & \xrightarrow{\iota_Y} & \overline{Y} \end{array}$$

such that the morphism $\iota_X\colon X \hookrightarrow \overline{X}$ is a good compactification of S-schemes, and such that $\overline{X}$ is finitely presented over S.

Proof. This is a consequence of Lemmas 2.5 and 2.9. □

Given a compactification of S-schemes $X \hookrightarrow \overline{X}$, we would like to control the fibrewise dimension of $\overline{X}$ in terms of the fibrewise dimension of X. In general, the fibrewise dimension of $\overline{X}$ may be higher than that of X (see for instance [DHKY23, Example 2.2]). However, if $\overline{X}$ is flat over S, we have the following result.

Lemma 2.11. *Let S be a scheme, $d \geqslant 0$ be an integer, and $X \hookrightarrow \overline{X}$ be a compactification of S-schemes such that $\overline{X}$ is flat and finitely presented over S. If X is S-fibrewise of dimension d (resp. S-fibrewise of pure dimension d), then the same holds for $\overline{X}$.*

Proof. The S-fibrewise dimension of the proper, flat, finitely presented S-scheme $\overline{X}$ is locally constant by [Sta25, Tag 0D4J], so the claim follows by checking the dimension of its fibres over each generic point of S (resp. by using [Sta25, Tag 02FZ]). □

Over a Prüfer domain, this flatness hypothesis is essentially always satisfied.

Lemma 2.12 (Compactifications over a Prüfer domain)**.** *Let S be the spectrum of a Prüfer domain, and $X \hookrightarrow \overline{X}$ be a schematically dense compactification of S-schemes. If X is flat over S, then so is $\overline{X}$, and X and $\overline{X}$ are then finitely presented over S.*

Proof. The scheme S being the spectrum of an integral domain, the flat S-scheme X is in particular torsionfree over S. The open immersion $\iota\colon X \hookrightarrow \overline{X}$ being schematically dense, the natural map of sheaves $\mathcal{O}_{\overline{X}} \to \iota_*\mathcal{O}_X$ is injective, hence the $\mathcal{O}_S$-module $\mathcal{O}_{\overline{X}}$ is also torsionfree. Because S is the spectrum of a Prüfer domain, this implies that the S-scheme $\overline{X}$ is flat (Remark 2.2 (2)). The schemes X and $\overline{X}$ are then flat and of finite type over the spectrum of an integral domain S, hence they are finitely presented over S ([RG71, première partie, corollaire 3.4.7]). □

Over a more general base, it may be hard to find a flat compactification. However, at least in the smooth case, one can have some control on the fibrewise dimension of the compactification (Corollary 2.14).

Lemma 2.13. *Let S be a qcqs scheme, and $X \hookrightarrow \overline{X}$ be a good compactification of finitely presented S-schemes where X is S-fibrewise of constant dimension. Then, the schematic closure $\overline{Z}$ inside $\overline{X}$ of every closed subscheme $Z \hookrightarrow X$ that is S-fibrewise of codimension one is S-fibrewise of constant dimension. Moreover, the S-scheme $\overline{Z}$ can be ensured to be finitely presented if S has a stably noetherian topological space.*

Proof. For every point $s \in S$, we have the equalities $\dim(Z_s) = \dim(X_s) - 1$ and $\dim(X_s) = \dim(\overline{X}_s)$ (Lemma 2.5), hence it suffices to prove that $\dim(\overline{Z}_s) = \dim(\overline{X}_s) - 1$. If it were not true, then $\overline{Z}_s$ would contain the generic point of an irreducible component of $\overline{X}_s$ of maximum dimension. Because the compactification $X \hookrightarrow \overline{X}$ is a good compactification, such a generic point necessarily lies in X_s, hence also in Z_s, which is in contradiction with the fact that $\dim(Z_s) < \dim(X_s)$. So the compactification $Z \hookrightarrow \overline{Z}$ is indeed a compactification by a finitely presented S-scheme fibrewise of constant dimension. The second claim is a consequence of Remark 2.7 (2). □

In the following results, given an S-scheme X fibrewise of constant dimension, we want to ensure that Zariski-locally around a point $x \in X$, the S-scheme X remains fibrewise of constant dimension over its image in S. If $x \in X$ lies over the point $s \in S$, a sufficient condition for this is that x lies in an irreducible component of maximum dimension of X_s.

Corollary 2.14. *Let S be local scheme, and X be a smooth S-scheme fibrewise of constant dimension. Then for every point $x \in X$ over the closed point $s \in S$ and such that x lies in an irreducible component of maximum dimension of X_s, there exists an affine Zariski neighbourhood $x \in X' \subseteq X$ that admits a compactification of S-schemes $X' \hookrightarrow \overline{X'}$ such that the S-scheme $\overline{X'}$ is fibrewise of constant dimension. Moreover, the S-scheme $\overline{X'}$ can be ensured to be finitely presented if S has a stably noetherian topological space.*

Proof. The smooth S-scheme X can be expressed, Zariski-locally around $x \in X$, and for some integer $N \geqslant 0$, as an effective Cartier divisor X' of an open subscheme of $\mathbb{A}^N_S$ ([SGA1, exposé II, théorème 4.10 (ii)] and [Sta25, Tag 02GT]). Any Zariski neighbourhood of a point of $\mathbb{A}^N_S$ which lies over the closed point $s \in S$ and inside an irreducible component of maximum dimension of $\mathbb{A}^N_s$ is S-fibrewise dense inside $\mathbb{A}^N_S$, hence its inclusion inside $\mathbb{P}^N_S$ is a good compactification of S-schemes. The first claim is then a consequence of Lemma 2.13, and the second claim is a consequence of Remark 2.7 (2). □

We prove now that the existence of a compactification fibrewise of constant dimension implies the existence of a good compactification. Following [GLL15, Definition 0.3], an affine scheme S is said to be *pictorsion* if the Picard group $\mathrm{Pic}(S')$ of any finite S-scheme S' is torsion. Examples of pictorsion schemes include all semilocal schemes, as well as the spectrum of the ring of integers of a number field ([GLL15, Definition 0.2 and Lemma 8.10 (2)]).

Proposition 2.15. *Let S be a pictorsion scheme (e.g., the spectrum of a semilocal ring), and X be a finitely presented S-scheme fibrewise of positive dimension d which admits a compactification by a finitely presented S-scheme $\overline{X}$ fibrewise of dimension d. Then for every closed point $x \in X$ lying over a closed point $s \in S$ such that x lies in an irreducible component of maximum dimension of X_s, there exist a Nisnevich neighbourhood $x \in X' \to X$, a smooth S-scheme T fibrewise of dimension $d-1$, a closed immersion $W \hookrightarrow \mathbb{P}^1_T$ such that W is finite over T, and a finite morphism of S-schemes $\varphi\colon X' \to \mathbb{P}^1_T \setminus W$.*

Proof. By [GLL15, Theorem 8.1], where we use that S is a pictorsion scheme and that $\overline{X}$ is a projective S-scheme of relative dimension d, there exists a finite S-morphism $f\colon \overline{X} \to \mathbb{P}^d_S$. Let $\widetilde{\mathbb{P}}^d_S$ be the blowup of $\mathbb{P}^d_S$ along any S-section which does not contain $f(x) \in \mathbb{P}^d_S$, $\widetilde{X}$ be the strict transform of $\overline{X}$, and replace X by its intersection with the complement of the blowup locus of $\overline{X}$. In particular, there are natural morphisms of S-schemes $\widetilde{X} \to \widetilde{\mathbb{P}}^d_S$ and $\widetilde{\mathbb{P}}^d_S \to \mathbb{P}^{d-1}_S$, and we denote by $g\colon \widetilde{X} \to \mathbb{P}^{d-1}_S$ the composite projective morphism, which is of relative dimension one.

We now use this morphism $g\colon \widetilde{X} \to \mathbb{P}^{d-1}_S$ to construct a finite morphism of S-schemes $\varphi\colon \widetilde{X}_T \to \mathbb{P}^1_T$, where T is a Nisnevich neighbourhood of $t := g(x) \in \mathbb{P}^{d-1}_S$, satisfying that $\varphi\colon x \mapsto \infty_T$ and that $(\widetilde{X}_T \setminus X_T) \cap \varphi^{-1}(\infty_T) = \varnothing$. Given such a morphism, the closed subscheme $W := \varphi(\widetilde{X}_T \setminus X_T)$ of $\mathbb{P}^1_T$ is then affine (because it is contained in $\mathbb{A}^1_T$) and proper over T (because $\mathbb{P}^1_T$ is), hence it is finite over T ([Sta25, Tag 01WN]). The S-scheme $X' := X_T \times_{\mathbb{P}^1_T} (\mathbb{P}^1_T \setminus W)$ is then a Nisnevich neighbourhood $x \in X$, and admits a finite map to $\mathbb{P}^1_T \setminus W$ ([Sta25, Tag 01WL]), hence the desired result.

To construct this morphism φ Nisnevich-locally around $t \in \mathbb{P}^{d-1}_S$, all the schemes involved being finitely presented over S, we can assume that T is the henselisation of the local scheme of $\mathbb{P}^{d-1}_S$ at t and, replacing X and $\widetilde{X}$ by their corresponding base change, that we have a projective morphism $g\colon \widetilde{X} \to T$ of relative dimension one. We then use [GLL15, Theorem 5.1], for the affine scheme $S := T$, the projective T-scheme $X := \widetilde{X}$, and for a very ample line bundle $\mathscr{L}$ relative to $\widetilde{X} \to T$, to construct the desired finite morphism $\varphi\colon \widetilde{X} \to \mathbb{P}^1_T$. More precisely, first take $C := \{x\}$, $m := 1$, $F_1 := \widetilde{X}$, and $A := \varnothing$, to produce a section $s_1 \in \mathrm{H}^0(\widetilde{X}, \mathscr{L}^{\otimes n})$, for some integer $n \geqslant 1$, whose vanishing hypersurface $H_1 \subseteq \widetilde{X}$ contains x and is finite over T. Note here that the hypotheses needed to apply [GLL15, Theorem 5.1] are indeed satisfied: x is a closed point of X which lies over a closed point of S, so $\{x\}$ is a closed subscheme of $\widetilde{X}$,[4] which moreover

[4] The closed subscheme $\{x\}$ can be ensured to be finitely presented over S if S has a stably noetherian topological space

does not contain any irreducible subscheme of any T-fibre of $\widetilde{X}$. By [Sta25, Tag 04GG (10)], where we use that T is a henselian local scheme, the finite T-scheme H_1 is a finite product of local schemes. To ensure that $(\widetilde{X} \setminus X) \cap H_1 = \varnothing$ (but still that $x \in H_1$), replace H_1 by the product of those local schemes whose closed point lies in X. Apply again [GLL15, Theorem 5.1], this time for $C := \varnothing$, $m := 0$, and $A := (H_1)_t$ (which is indeed a finite set of points), to produce a section $s_0 \in \mathrm{H}^0(\widetilde{X}, \mathscr{L}^{\otimes nn'})$, for some integer $n' \geqslant 1$, whose vanishing hypersurface $H_0 \subseteq \widetilde{X}$ does not intersect $(H_1)_t$. By construction, the intersection $H_0 \cap H_1$ is proper (hence closed) over the local scheme T, and its closed fibre is empty, hence it is empty. These hypersurfaces $H_0, H_1 \subseteq \widetilde{X}$ induce a morphism $\varphi\colon \widetilde{X} \to \mathbb{P}^1_T$ which sends x to ∞_T, and satisfies that $(\widetilde{X} \setminus X) \cap \varphi^{-1}(\infty_T) = \varnothing$. This morphism φ is necessarily proper ([Sta25, Tag 01W6]), so its fibres are proper ([Sta25, Tag 01W4]) and affine (because it is contained in the non-vanishing locus of one of s_0 or s_1, which is affine), hence they are finite ([Sta25, Tag 01WN]). The morphism φ is then proper and quasi-finite, hence it is finite ([Sta25, Tag 02LS]), as desired. □

Theorem 2.16 (Good compactification)**.** *Let S be a henselian local scheme which is stably coherent and has a stably noetherian topological space, and X be a finitely presented S-scheme fibrewise of constant dimension. If X admits a compactification by a finitely presented S-scheme fibrewise of constant dimension, then for every point $x \in X$ over the closed point $s \in S$ such that x lies in an irreducible component of maximum dimension of X_s, there exists a Nisnevich neighbourhood $x \in X' \to X$ that admits a good compactification of S-schemes $X' \hookrightarrow \overline{X'}$.*

Proof. We prove the result by induction on the fibrewise dimension d of the S-scheme X.

If $d = 0$, then the morphism of relative dimension zero $X \to S$ is locally quasi-finite ([Sta25, Tag 0397]). By Hensel's lemma [Gro67, théorème 18.5.11 $a) \Rightarrow c'')$], where we use that S is a henselian local scheme, that $X \to S$ is quasi-finite at $x \in X$, and that $x \in X$ lies over the closed point of S, there exists a Zariski neighbourhood $x \in X' \subseteq X$ which is finite over S. In particular, the S-scheme X' is projective ([Sta25, Tag 0B3I]), and we can take $\overline{X'} := X'$.

Assume now that $d \geqslant 1$ and that the result is proved for S-schemes fibrewise of dimension $d-1$. By Proposition 2.15, where we use that S is a local scheme, there exist a Nisnevich neighbourhood $x \in X' \to X$, a smooth S-scheme T fibrewise of dimension $d-1$, a closed immersion $W \hookrightarrow \mathbb{P}^1_T$ such that W is finite over T, and a finite morphism of S-schemes $\varphi\colon X' \to \mathbb{P}^1_T \setminus W$. By Corollary 2.14, the smooth S-scheme T admits, Nisnevich-locally around $t \in T$, a compactification by a finitely presented S-scheme fibrewise of constant dimension. So, by the induction hypothesis, the S-scheme T admits a good compactification of S-schemes $T \hookrightarrow \overline{T}$. By Remark 2.4 (3), the induced morphism $\mathbb{P}^1_T \hookrightarrow \mathbb{P}^1_{\overline{T}}$ is a good compactification of finitely presented S-schemes fibrewise of dimension d. The open immersion of S-schemes $\mathbb{P}^1_T \setminus W \hookrightarrow \mathbb{P}^1_T$ is moreover fibrewise dense, so the composite morphism $\mathbb{P}^1_T \setminus W \hookrightarrow \mathbb{P}^1_{\overline{T}}$ is a good compactification of S-schemes. By Corollary 2.10, this implies that the S-scheme X' admits a good compactification by a finitely presented S-scheme $\overline{X'}$, which concludes the induction. □

Corollary 2.17 (Good compactification over a valuation ring)**.** *Let S be the spectrum of a henselian valuation ring of finite rank, and X be a finitely presented, flat S-scheme fibrewise of constant dimension. Then, for every point $x \in X$ over the closed point $s \in S$ such that x lies in an irreducible component of maximum dimension of X_s, there exists a Nisnevich neighbourhood $x \in X' \to X$ that admits a good compactification of S-schemes $X' \hookrightarrow \overline{X'}$.*

Proof. The scheme S is stably coherent and has a stably noetherian topological space (Example 2.8), and the S-scheme X admits a compactification by a finitely presented S-scheme fibrewise of constant dimension (Lemmas 2.11 and 2.12), so the result is a consequence of Theorem 2.16. □

The following result was also obtained by Deshmukh–Hogadi–Kulkarni–Yadav [DHKY23, Theorem 1.2] in the case of a noetherian base of finite dimension.

(Remark 2.7 (2)). More generally, one can slightly modify the noetherian approximation part of the proof of [GLL15, Theorem 5.1] to see that the finitely presented hypothesis is not necessary in the case where the closed subscheme C is a finite set of points.

Corollary 2.18 (Good compactification for local complete intersections)**.** *Let S be a henselian local scheme which is stably coherent and has a stably noetherian topological space (e.g., a noetherian henselian local scheme), and X be a local complete intersection over S in the sense of [Sta25, Tag 069F]. Then, for every point $x \in X$ over the closed point $s \in S$ such that x lies an irreducible component of maximum dimension of X_s, there exists a Nisnevich neighbourhood $x \in X' \to X$ that admits a good compactification of S-schemes $X' \hookrightarrow \overline{X'}$.*

Proof. If X is Zariski-locally around $x \in X$ of the form $\mathbb{A}^N_S$ for some integer $N \geqslant 0$, then it is a consequence of the fact that the morphism $\mathbb{A}^N_S \hookrightarrow \mathbb{P}^N_S$ is a good compactification of S-schemes. The result is then a consequence of using iteratively first Lemma 2.13, and then Theorem 2.16 and Corollary 2.14. ☐

Proposition 2.19. *Let S be an affine scheme, $N \geqslant 0$ be an integer, $Z \hookrightarrow \mathbb{A}^N_S$ be a closed immersion of S-schemes, and $\overline{Z}$ be the schematic closure of Z inside $\mathbb{P}^N_S$. Then for every principal closed immersion of S-schemes $Z \hookrightarrow X$, there exists a compactification of S-schemes $X \hookrightarrow \overline{X}$ such that the schematic closure morphism of Z inside $\overline{X}$ is the compactification $Z \hookrightarrow \overline{Z}$.*

Proof. The closed immersion $Z \hookrightarrow \mathbb{A}^N_S$ lifts to a morphism $f\colon X \to \mathbb{A}^N_S$. If the morphism f is a locally closed immersion, take $\overline{X}$ to be the schematic closure of X inside $\mathbb{P}^N_S$ via the open immersion $\mathbb{A}^N_S \hookrightarrow \mathbb{P}^N_S$. In general, it suffices to produce a locally closed immersion of S-schemes $g\colon X \hookrightarrow \mathbb{A}^M_S$, for some integer $M \geqslant 0$, such that $g|_Z = 0$. Indeed, given such a locally closed immersion, take $\overline{X}$ to be the schematic closure of X inside $\mathbb{P}^{N+M}_S$ via the morphism

$$X \xhookrightarrow{(f,g)} \mathbb{A}^{N+M}_S \hookrightarrow \mathbb{P}^{N+M}_S,$$

so that the scheme theoretic image of Z inside $\mathbb{A}^{N+M}_S$ lies inside $\mathbb{A}^N_S$, hence its schematic closure inside $\overline{X}$ is again naturally identified with $\overline{Z}$.

To produce the locally closed immersion $g\colon X \to \mathbb{A}^M_S$ such that $g|_Z = 0$, recall that S, X, and Z are affine, and that the closed immersion $Z \hookrightarrow X$ is assumed to be principal. Let $R := \mathcal{O}_S(S)$, $A := \mathcal{O}_X(X)$, and $t \in A$ such that $Z = V(t)$. The R-algebra A is of finite type, so there exists a surjection $R[x_1, \dots, x_M] \twoheadrightarrow A$, where we denote by t_i the image of x_i. Let $g\colon X \to \mathbb{A}^M_S$ be the morphism defined by $x_i \mapsto t \cdot t_i$. By construction, the morphism g is zero on Z (because $Z = V(t)$) and is an immersion (because t is invertible on $X \setminus Z$), which concludes the proof. ☐

Remark 2.20. Given an affine scheme S and a good compactification of finitely presented S-schemes $X \hookrightarrow \overline{X}$ such that $\overline{X} \setminus X$ is finitely presented over S and X is of S-fibrewise dimension d, then for every point $x \in X$, there exists an affine Zariski neighbourhood $x \in X' \subseteq X$ such that the open immersion $X' \hookrightarrow \overline{X}$ is a good compactification of S-schemes, and such that there exist an integer $N \geqslant 0$ and a closed immersion of S-schemes $X' \hookrightarrow \mathbb{A}^N_S$ satisfying that $\overline{X}$ is the schematic closure of X' inside $\mathbb{P}^N_S$. Indeed, if $d = 0$, then the good compactification $X \hookrightarrow \overline{X}$ is an isomorphism of S-schemes, and we can take $X' := X$. Otherwise, given a closed immersion of S-schemes $\overline{X} \hookrightarrow \mathbb{P}^n_S$, then, by [GLL15, Theorem 5.1], where we take $C := \overline{X} \setminus X$, $m := 1$, $F_1 := X$, and $A := \{x\}$, there exists a hypersurface $H \subseteq \mathbb{P}^n_S$ which contains $\overline{X} \setminus X$, and avoids both $x \in X$ and the generic points of the irreducible components of the fibres X_s of positive dimension. Taking the Veronese embedding $\mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$ associated to the hypersurface $H \subseteq \mathbb{P}^n_S$, the S-scheme $X' := X \setminus (X \cap H)$ then satisfies the desired requirements.

Corollary 2.21. *Let S be an affine scheme, X be a finitely presented S-scheme fibrewise of constant dimension, and $Z \hookrightarrow X$ be a closed immersion of S-schemes. Then for every point $x \in Z$ lying over $s \in S$ such that x lies in an irreducible component of maximum dimension of Z_s and such that $Z \hookrightarrow X$ is principal and S-fibrewise of positive codimension in a Zariski neighbourhood of $x \in X$, there exist affine Nisnevich neighbourhoods $s \in S' \to S$ and $x \in X' \to X \times_S S'$ such that the closed immersion of S'-schemes $Z \times_X X' \hookrightarrow X'$ admits a good compactification, in each of the following cases:*

(i) S is the spectrum of a Prüfer ring of finite Krull dimension;

(ii) S is stably coherent and has a stably noetherian topological space (e.g., S is noetherian), and X is a local complete intersection over S.

Proof. The statement is Nisnevich-local around the point $s \in S$, and all the objects involved are finitely presented, so we can assume that S is a henselian local scheme. By Corollaries 2.17 and 2.18, there exists a Nisnevich neighbourhood $x \in Z' \to Z$ that admits a good compactification of S-schemes $Z' \hookrightarrow \overline{Z'}$. We can also assume that, Zariski-locally around $x \in Z$, this Nisnevich neighbourhood $x \in Z' \to Z$ is a standard one [Sta25, Tag 00UE] and that $Z \hookrightarrow X$ is a principal closed immersion, hence in particular that $Z' \to Z$ is the restriction to Z of a Nisnevich neighbourhood $x \in X' \to X$. By Remark 2.20, we can further assume that Z' and X' are affine, and that there exists a closed immersion $Z' \hookrightarrow \mathbb{A}^N_S$, for some integer $N \geqslant 0$, such that $\overline{Z'}$ is the schematic closure of Z' inside $\mathbb{P}^N_S$. By Proposition 2.19, the good compactification $Z' \hookrightarrow \overline{Z'}$ then promotes to a good compactification of the closed immersion $Z' \hookrightarrow X'$. ☐

The rest of the section is dedicated to constructing a geometric presentation (Theorem 2.24) following the approach developed in [Kun23]. A key input is the following result.

Lemma 2.22 ([Čes22a])**.** *Let T be an affine scheme, X be a finitely presented, flat, affine T-scheme with Cohen–Macaulay fibres of pure dimension one (e.g., a smooth affine T-scheme fibrewise of pure dimension one), and $Z \hookrightarrow X$ be a finitely presented closed immersion of T-schemes. If the T-scheme Z is finite, then for every point $x \in X$ lying over $t \in T$, there exist*

(a) an affine Nisnevich neighbourhood $x \in X' \to X$ whose image contains Z,

(b) an affine Zariski neighbourhood $t \in T' \subseteq T$,

(c) and an étale morphism $\psi\colon X' \to \mathbb{A}^1_{T'}$

such that ψ maps $Z' := Z \times_X X'$ isomorphically onto a closed subscheme $\psi(Z') \subseteq \mathbb{A}^1_{T'}$, which satisfies that $Z' \cong \psi(Z') \times_{\mathbb{A}^1_{T'}} X'$.

Proof. If the scheme T is semilocal, this is [Čes22a, Lemmas 6.1 and 6.3], where we take $R := \mathcal{O}_T(T)$, $C := X \sqcup \mathbb{A}^1_T$, and $Z := Z \sqcup T$. Indeed, these results ensure the existence of a suitable étale morphism $X' \to X$, and this morphism may be arranged to be a Nisnevich neighbourhood of $x \in X$ by a slight modification of the proof of [Čes22a, Lemma 6.1]. More precisely, the scheme X' is constructed as an extension of affine schemes $X' \to X$ generated by a polynomial $f(z) \in \mathcal{O}_X(X)[z]$; as in the proof of loc. cit., this polynomial can be chosen to restrict to a polynomial $f_x(z) \in z\mathcal{O}_{X,x}[z]$ in $\mathcal{O}_{X,x}[z]$. This factor z of $f_x(z)$ then induces the desired lift of $x \in X$ in X'.

In general, the statement is Zariski-local around the point $t \in T$, and all the objects involved are finitely presented, so the previous case implies the desired result in a Zariski neighbourhood $T' \subseteq T$ of $t \in T$. ☐

Proposition 2.23 ([Kun25])**.** *Let S be a stably coherent affine scheme, X be a smooth S-scheme fibrewise of pure dimension d, $X \hookrightarrow \overline{X}$ be a compactification of S-schemes where $\overline{X}$ is finitely presented, and $\overline{Z} \hookrightarrow \overline{X}$ be a closed immersion of S-schemes such that $\overline{Z} \setminus (\overline{Z} \cap X)$ is S-fibrewise of codimension $\geqslant 2$ inside $\overline{X}$. Then for every point $x \in X$, there exist*

(a) an affine Zariski neighbourhood $x \in X' \subseteq X$,

(b) an affine open subscheme $T \subseteq \mathbb{A}^{d-1}_V$,

(c) and a smooth morphism $X' \to T$ fibrewise of pure dimension one

such that $\overline{Z} \cap X'$ is finite over T.

Proof. This is [Kun25, Theorem 2.15 (b)], where we take $R := \mathcal{O}_S(S)$, $X := \overline{X}$, $W := X$, $Y := \overline{Y}$, $n := 1$, and $x_1 := x$, and where the integral domain hypothesis is used in the proof only when $n > 1$. ☐

Theorem 2.24. *Let S be a stably coherent affine scheme, X be a smooth S-scheme fibrewise of pure dimension d, and $Z \hookrightarrow X$ be a finitely presented closed immersion of S-schemes which is S-fibrewise of positive codimension. If $Z \hookrightarrow X$ admits a good compactification in the sense of Definition 2.3 (3), then for every point $x \in X$, there exist*

(a) an affine Nisnevich neighbourhood $x \in X' \to X$,

(b) an affine open subscheme $T \subseteq \mathbb{A}^{d-1}_S$,

(c) and an étale morphism $\psi\colon X' \to \mathbb{A}^1_T$

such that, for $Z' := Z \times_X X'$*, the morphism* $\psi|_{Z'}\colon Z' \to \mathbb{A}^1_T$ *is a closed immersion, the image* $\psi(Z')$ *is finite over* T*, and* $Z' \cong \psi(Z') \times_{\mathbb{A}^1_T} X'$.

Proof. By assumption, there exists a dense open immersion of S-schemes $X \hookrightarrow \overline{X}$ where $\overline{X}$ is a projective S-scheme, and such that the schematic closure morphism $Z \hookrightarrow \overline{Z}$ inside $\overline{X}$ is a good compactification. In particular, for every point $s \in S$ such that the fibre $\overline{Z}_s$ is non-empty, this implies that $\dim(\overline{Z}_s \setminus Z_s) < \dim(\overline{Z}_s) = \dim(Z_s)$. We also know that Z is S-fibrewise of positive codimension inside X, so that $\dim(\overline{Z}_s) < d$. The S-scheme $\overline{X}$ is fibrewise of dimension at least d, so this implies that $\overline{Z} \setminus Z$ is S-fibrewise of codimension $\geqslant 2$ inside $\overline{X}$. Hence, by Proposition 2.23, there exist an affine Zariski neighbourhood $x \in X_0 \subseteq X$, an affine open subscheme $T_0 \subseteq \mathbb{A}^{d-1}_S$, and a smooth morphism $X_0 \to T_0$ fibrewise of pure dimension one such that $\overline{Z}_0 := \overline{Z} \cap X_0$ is finite over T_0. It remains now to produce, at least Nisnevich-locally, an étale morphism $\psi : X_0 \to \mathbb{A}^1_{T_0}$ satisfying that $\psi|_{Z_0} : Z_0 \to \mathbb{A}^1_{T_0}$ is a closed immersion and that $Z_0 \cong \psi(Z_0) \times_{\mathbb{A}^1_{T_0}} X_0$. By Lemma 2.22 applied to the finitely presented closed immersion of T_0-schemes $Z_0 \hookrightarrow X_0$, there exist an affine Nisnevich neighbourhood $x \in X' \to X_0$ whose image contains Z_0, an affine open subscheme $T \subseteq T_0$, and an étale morphism $\psi\colon X' \to \mathbb{A}^1_T$ such that ψ maps $Z' := Z_0 \times_{X_0} X'$ isomorphically onto a closed subscheme $\psi(Z') \subseteq \mathbb{A}^1_T$, which satisfies that $Z' \cong \psi(Z') \times_{\mathbb{A}^1_T} X'$. By construction, the scheme $\psi(Z')$ is moreover finite over T, hence the desired result. □

Corollary 2.25 (Presentation lemma)**.** *Let* S *be a scheme,* X *be a smooth* S*-scheme, and* $Z \hookrightarrow X$ *be a closed immersion of* S*-schemes. Then for every point* $x \in X$ *lying over a point* $s \in S$ *such that* $\dim(Z_s) < \dim(X_s)$*, there exist*

(a) affine Nisnevich neighbourhoods $s \in S' \to S$ *and* $x \in X' \to X \times_S S'$,

(b) an affine open subscheme $T \subseteq \mathbb{A}^{d-1}_{S'}$,

(c) and an étale morphism $\psi\colon X' \to \mathbb{A}^1_T$

such that, for $Z' := Z \times_X X'$*, the morphism* $\psi|_{Z'}\colon Z' \to \mathbb{A}^1_T$ *is a closed immersion, the image* $\psi(Z')$ *is finite over* T*, and* $Z' \cong \psi(Z') \times_{\mathbb{A}^1_T} X'$*. In particular, the commutative diagram of* T*-schemes*

$$\begin{array}{ccc} X' \setminus Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \psi \\ \mathbb{A}^1_T \setminus \psi(Z') & \hookrightarrow & \mathbb{A}^1_T \end{array}$$

is a Nisnevich square, i.e., it is a pullback square of schemes, $\mathbb{A}^1_T \setminus \psi(Z') \hookrightarrow \mathbb{A}^1_T$ *is an open immersion,* ψ *is an étale morphism, and* $\psi(Z') \times_{\mathbb{A}^1_T} X' \to \psi(Z')$ *is an isomorphism.*

Proof. The claim is Nisnevich-local around $s \in S$ and $x \in X$, so we can assume that S is a henselian local scheme with closed point s, that X is S-fibrewise of pure dimension d, and that x lies in an irreducible component of maximum dimension of Z_s. The scheme S is a filtered colimit of noetherian henselian local schemes (which are in particular stably coherent and have a stably noetherian topological space), and the claim is stable under base change on S, so we can further assume that S is stably coherent and has a stably noetherian topological space. By [GLL15, Theorem 5.1], up to replacing Z by a bigger closed subscheme of X, we can assume that the closed immersion $Z \hookrightarrow X$ is principal (and still that $\dim(Z_s) < \dim(X_s)$). Then, by Corollary 2.21 (*ii*), there exists a Nisnevich neighbourhood $x \in X' \to X$ such that the closed immersion of S-schemes $Z \times_X X' \hookrightarrow X'$ admits a good compactification. The desired result is then a consequence of Theorem 2.24. □

Remark 2.26. When S is the spectrum of a Dedekind domain with infinite residue fields, the previous result was proved by Schmidt–Strunk ([SS18, Theorem 2.4]). To the best of our knowledge, the previous result is new when S is the spectrum of a Dedekind domain with not necessarily infinite residue fields (*e.g.*, when $S = \mathrm{Spec}(\mathbb{Z})$).

3 Gersten's injectivity in mixed characteristic

In this section, we apply the presentation lemma of the previous section to prove a Nisnevich-local version of Gersten's injectivity for smooth schemes over not necessarily discrete valuation rings (Corollary 3.13). This result applies in particular for $\mathbb{A}^1$-invariant Nisnevich sheaves (Remark 3.9) and for non-$\mathbb{A}^1$-invariant motivic spectra in the sense of [AHI24] (Corollary 3.15), and will be the main new ingredient in the proofs of Theorems A and B.

We first prove a Gersten's injectivity statement in the context of Nisnevich sheaves of pointed sets (Theorem 3.5). The main property that we ask for these Nisnevich sheaves is the Horrocks principle (Definition 3.1), whose terminology is inspired by the literature on G-torsors (see for instance [Čes22b, Proposition 2.1.5]), and which is a non-commutative variant of the axiom "SUB2" of [CTHK97] (see Definition 3.6). We then apply this result to the homotopy groups of spectra-valued Nisnevich sheaves to prove our main Gersten's injectivity result (Corollary 3.13).

For the rest of this section, given a scheme X, we denote by ∞_X, j_X, π_X and $\bar{\pi}_X$ the canonical morphisms of schemes appearing in the following commutative diagram.

$$\begin{array}{ccccc} \mathbb{A}^1_X & \xhookrightarrow{j_X} & \mathbb{P}^1_X & \xhookleftarrow{\infty_X} & X \\ & {\scriptstyle \pi_X}\searrow & \downarrow{\scriptstyle \bar{\pi}_X} & /\!/ & \\ & & X & & \end{array} \qquad (*)$$

Definition 3.1 (Horrocks principle)**.** Given a scheme S, a functor $F\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Set}_*$ is said to satisfy the *Horrocks principle* if for any qcqs S-scheme X, every class $[\sigma] \in F(\mathbb{P}^1_X)$ that trivialises at ∞ (*i.e.*, $\infty_X^*[\sigma] = [*]$) also trivialises on $\mathbb{A}^1_X$ (*i.e.*, $j_X^*[\sigma] = [*]$).

Our argument to establish Gersten's injectivity follows the standard techniques axiomatised by Colliot-Thélène–Hoobler–Kahn [CTHK97], where we replace the reduction to the case of fields by a reduction to the case of henselian valuation rings. The following result is a crucial step in proving our main Gersten's injectivity result (Theorem 3.5), and is the only point in our argument where we use the presentation lemma (Corollary 2.25).

Theorem 3.2. *Let P be a Prüfer ring, R be the henselisation of an essentially smooth local P-algebra, and $s \in \mathrm{Spec}(P)$ be the image of the closed point of $\mathrm{Spec}(R)$. Then the fibre $\mathrm{Spec}(R)_s$ of $\mathrm{Spec}(R)$ over s is irreducible. Moreover, for every finitary Nisnevich sheaf $F\colon \mathrm{Sch}_P^{\mathrm{qcqs,op}} \to \mathrm{Set}_*$ satisfying the Horrocks principle in the sense of Definition 3.1, we have that*

$$\ker(F(R) \to F(R_\eta)) = \{*\},$$

where $\eta \in \mathrm{Spec}(R)$ is the generic point of the irreducible scheme $\mathrm{Spec}(R)_s$.

Proof. The claims of interest are insensitive to replacing the Prüfer ring P by the henselisation of its localisation at s. In particular, we can assume that P is a henselian valuation ring with maximal ideal s. The scheme $\mathrm{Spec}(R)$ is local, thus, so is the scheme $\mathrm{Spec}(R)_s$. The essential smoothness hypothesis on the P-algebra R ensures that the scheme $\mathrm{Spec}(R)_s$ is moreover regular. Consequently, the scheme $\mathrm{Spec}(R)_s$ is irreducible ([Sta25, Tag 00NP]). To prove the second claim, because the presheaf F commutes with filtered colimits of rings, we can further reduce to the case where the valuation ring P is of finite rank (see for instance [Kun24, Lemma 2.5 (b)]).

Let $[\sigma] \in F(R)$ be a class whose image in $F(R_\eta)$ vanishes. We want to prove that $[\sigma]$ itself vanishes. For the rest of this proof, given a closed immersion of P-schemes $Z \hookrightarrow X$, we denote by $F_Z(X)$ the pointed set $\ker(F(X) \to F(X \setminus Z))$. The presheaf F commutes with filtered colimits of rings, so there exist a smooth affine P-scheme X and a closed immersion of P-schemes $Z \hookrightarrow X$ such that R is the henselisation of the local ring of X at some point $x \in X$ lying over $s \in \mathrm{Spec}(P)$ and such that the class $[\sigma] \in F(R)$ is the image of a class $[\sigma] \in F(X)$ that vanishes in $F(X \setminus Z)$, *i.e.*, $[\sigma] \in F(R)$ is the image of a class $[\sigma] \in F_Z(X)$.

To prove the desired result, it then suffices to produce a Nisnevich neighbourhood $x \in X' \to X$ which satisfies, for $Z' := Z \times_X X' \hookrightarrow X'$, that the pullback class $[\sigma'] \in F_{Z'}(X')$ vanishes. If R is of relative dimension zero over P, then the local ring $\mathcal{O}_{X,x}$ is étale over P, and is even a valuation ring with maximal ideal x ([Sta25, Tag 0ASJ]). In particular, the local ring R has maximal ideal η, and $R = R_\eta$. We assume now that R is of positive relative dimension over P. By [Kun24, Lemma 2.13 (2)], where we use that the valuation ring P is of finite rank, we can then further assume that $\dim(Z_s) < \dim(X_s)$.

Therefore, by the presentation lemma proved in the previous section over the henselian local scheme S (Corollary 2.25), there exist a smooth S-scheme T, a Nisnevich neighbourhood $X' \to X$ of x, and a Nisnevich square

$$\begin{array}{ccc} X' \setminus Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \psi \\ \mathbb{A}^1_T \setminus \psi(Z') & \hookrightarrow & \mathbb{A}^1_T \end{array}$$

such that the induced morphism $Z' \xrightarrow{\cong} \psi(Z') \to T$ is finite. By Nisnevich excision, the natural map $F_{\psi(Z')}(\mathbb{A}^1_T) \to F_{Z'}(X')$ is an isomorphism, and it then suffices to prove that the class $[\sigma'] \in F_{\psi(Z')}(\mathbb{A}^1_T)$ vanishes. By finiteness of the map $\psi(Z') \to T$, the composite morphism $\psi(Z') \hookrightarrow \mathbb{A}^1_T \hookrightarrow \mathbb{P}^1_T$ is a closed immersion ([Sta25, Tags 01WN and 01W6]), hence the commutative diagram of T-schemes

$$\begin{array}{ccc} \mathbb{A}^1_T \setminus \psi(Z') & \hookrightarrow & \mathbb{A}^1_T \\ \downarrow & & \downarrow j_T \\ \mathbb{P}^1_T \setminus \psi(Z') & \hookrightarrow & \mathbb{P}^1_T \end{array}$$

is a Nisnevich square. Again, by Nisnevich excision, the natural map $F_{\psi(Z')}(\mathbb{P}^1_T) \to F_{\psi(Z')}(\mathbb{A}^1_T)$ is an isomorphism, hence there exists a class $[\tilde{\sigma}'] \in F(\mathbb{P}^1_T)$ whose image in $F(\mathbb{P}^1_T \setminus \psi(Z'))$ vanishes and such that $j_T^*[\tilde{\sigma}'] = [\sigma'] \in F(\mathbb{A}^1_T)$. By construction, the closed subscheme $\psi(Z') \subseteq \mathbb{P}^1_T$ does not meet the ∞-section of $\mathbb{P}^1_T$, so the map $\infty_T^* \colon F(\mathbb{P}^1_T) \to F(T)$ naturally factors through the map $F(\mathbb{P}^1_T) \to F(\mathbb{P}^1_T \setminus \psi(Z'))$. In particular the class $\infty_T^*[\tilde{\sigma}'] \in F(T)$ vanishes. By the Horrocks principle (Definition 3.1), the class $j_T^*[\tilde{\sigma}'] \in F(\mathbb{A}^1_T)$ then also vanishes, which concludes the proof. □

The final ingredient in the proof of Theorem 3.5 is Corollary 3.4, which is obtained as a consequence of the following result.

Lemma 3.3. *Let $\mathcal{V}$ be a subcategory of the category of commutative rings that is closed under localisations of rings at multiplicative subsets, and $F \colon \mathcal{V} \to \mathrm{Set}_*$ be a finitary Nisnevich sheaf. Then, the following are equivalent:*

(1) for every henselian valuation ring $V \in \mathcal{V}$ of finite rank, we have

$$\ker(F(V) \to F(\mathrm{Frac}(V))) = \{*\};$$

(2) for every semilocal Prüfer domain $P \in \mathcal{V}$, we have

$$\ker(F(P) \to F(\mathrm{Frac}(P))) = \{*\}.$$

Proof. Every henselian valuation ring is a semilocal Prüfer domain, so it suffices to prove that (1) ⇒ (2). Let P be a semilocal Prüfer domain. The presheaf F being finitary, we may assume by a limit argument that the Prüfer domain P is of finite Krull dimension ([Kun24, Lemma 2.5]). Letting $\mathrm{MaxSpec}(P)$ be the maximal spectrum of P, we prove the desired result by induction on the integer

$$d = \delta(P) := \sum_{\mathfrak{m} \in \mathrm{MaxSpec}(P)} \dim(P_{\mathfrak{m}}) < \infty.$$

If $d = 0$, then $P = \mathrm{Frac}(P)$, and the claim is true. We assume now that $d > 0$, and that the result is true for all semilocal Prüfer domains P' satisfying $\delta(P') < d$. Let $\mathfrak{m}$ be a maximal ideal of P, and let a be an element of P whose vanishing locus equals the subset $\{\mathfrak{m}\} \subseteq \mathrm{Spec}(P)$.[5] The ring $P[\frac{1}{a}]$ then

[5]Since P has finitely many prime ideals, such an a always exist by prime avoidance [Sta25, Tag 00DS].

satisfies $\delta(P[\frac{1}{a}]) = d-1$ (since $\mathrm{Spec}(P[\frac{1}{a}]) = \mathrm{Spec}(P) \setminus \{\mathfrak{m}\}$), and is a semilocal Prüfer domain ([Kun24, Lemma 2.2 (1)]). Consequently, by the induction hypothesis, it suffices to prove the equality

$$\ker(F(P) \to F(P[\tfrac{1}{a}])) = \{*\}.$$

By Nisnevich excision, the left-hand side of this equality is naturally identified with the pointed set $\ker(F(P^h_{\mathfrak{m}}) \to F(P^h_{\mathfrak{m}}[\frac{1}{a}]))$, where $P^h_{\mathfrak{m}}$ is the henselisation of P along the ideal $\mathfrak{m}$. The henselisation of a local ring of a Prüfer domain is a henselian valuation ring, so the result is a consequence of (1). □

Corollary 3.4. *Let $\mathcal{V}$ be a subcategory of the category of commutative rings that is closed under localisations of rings at multiplicative subsets, and $F\colon \mathcal{V} \to \mathrm{Set}_*$ be a finitary Nisnevich sheaf. If $F(V) = \{*\}$ for every henselian valuation ring $V \in \mathcal{V}$ of finite rank, then $F(P) = \{*\}$ for every semilocal Prüfer domain $P \in \mathcal{V}$.*

Proof. Let $P \in \mathcal{V}$ be a semilocal Prüfer domain. Any field is a henselian valuation ring of finite rank, so that $F(\mathrm{Frac}(P)) = \{*\}$, and in particular $F(P) = \ker(F(P) \to F(\mathrm{Frac}(P)))$. Lemma 3.3 and the assumption on henselian valuation rings of finite rank then imply the desired result. □

Theorem 3.5. *Let P be a Prüfer ring and $F\colon \mathrm{Sch}_P^{\mathrm{qcqs,op}} \to \mathrm{Set}_*$ be a finitary Nisnevich sheaf satisfying the following conditions:*

(i) the presheaf F satisfies the Horrocks principle in the sense of Definition 3.1;

(ii) for every henselian valuation ring V which is an ind-smooth P-algebra, we have $F(V) = \{*\}$.

Then, for every ind-smooth P-scheme X, we have $F(X) = \{*\}$.

Proof. The presheaf F being a finitary Nisnevich sheaf, we can assume that $X = \mathrm{Spec}(R)$, where R is the henselisation of an essentially smooth local P-algebra, and that P is a valuation ring of finite rank (see for instance [Kun24, Lemma 2.5]). By Theorem 3.2, the P-fibre of the closed point of $\mathrm{Spec}(R)$ admits a unique generic point $\eta \in \mathrm{Spec}(R)$, and the localisation $A := R_\eta$ satisfies that

$$\ker(F(R) \to F(A)) = \{*\},$$

hence it suffices to prove that $F(A) = \{*\}$. The local P-algebra A is moreover a semilocal Prüfer domain of finite Krull dimension ([Kun24, Lemma 2.13 (1)]). The hypothesis (ii) and Corollary 3.4 then imply that $F(A) = \{*\}$. □

Although Theorem 3.5 is stated in the generality of presheaves of pointed sets, our interest in this article lies primarily in presheaves of spectra. Accordingly, in the rest of this section, we explain how this result can be applied to certain spectra-valued presheaves, which we call *deflatable* (Definition 3.6). This condition essentially corresponds to the axiom “SUB2” of [CTHK97], and is equivalent to the one introduced in [EM23, Definition 6.7]. Note that any $\mathbb{A}^1$-invariant presheaf is deflatable (Remark 3.9), and so is any family of functors satisfying the $\mathbb{P}^1$-bundle formula (Lemma 3.12).

Definition 3.6 (Deflatability)**.** Given a scheme S, a functor $\mathscr{F}\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Sp}$ is called *deflatable* if for any qcqs S-scheme X, there exists a functorial equivalence between the maps

$$\mathscr{F}(\mathbb{P}^1_X) \underset{(\infty_X \circ \pi_X)^*}{\overset{j_X^*}{\rightrightarrows}} \mathscr{F}(\mathbb{A}^1_X),$$

where the maps ∞_X, j_X, and π_X are defined in ($*$).

Remark 3.7. In the previous definition, note that a functor $\mathscr{F}\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Sp}$ is deflatable as soon as the category of equivalences $\mathrm{Eq}(j^*, (\infty \circ \pi)^*)$ between the functors j^* and $(\infty \circ \pi)^*$ is nonempty. In particular, any such equivalence is not part of the data (see also [EM23, Remark 6.8]).

Remark 3.8. Let S be a scheme, and $\mathscr{F}\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Sp}$ be a deflatable functor. Then for every integer $i \in \mathbb{Z}$, the functor $\pi_i(\mathscr{F}(-))\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Set}_*$ satisfies the Horrocks principle in the sense of Definition 3.1.

Remark 3.9. Let S be a qcqs scheme, and $\mathscr{F}\colon \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Sp}$ be a functor. If $\mathscr{F}$ is $\mathbb{A}^1$-invariant, then $\mathscr{F}$ is deflatable. Indeed, in this case the natural map π^* is an equivalence and, for any section $s \in \mathbb{P}^1_S(S)\backslash\{\infty_S\}$, both j^* and ∞^* are $\mathbb{A}^1$-homotopy equivalent to s^*.

Definition 3.10 ($\mathbb{P}^1$-bundle formula)**.** Let S be a scheme. A family of presheaves $\mathscr{F}(i) : \mathrm{Sch}_S^{\mathrm{qcqs,op}} \to \mathrm{Sp}$, $i \in \mathbb{Z}$, equipped for every $i \in \mathbb{Z}$ with a map of functors $c_1 : \mathrm{Pic}(-) \to \mathrm{Hom}(\mathscr{F}(i-1)(-)[-2], \mathscr{F}(i)(-))$, is said to *satisfy the* $\mathbb{P}^1$*-bundle formula* if for every qcqs S-scheme X and every integer $i \in \mathbb{Z}$, the natural map

$$\bar{\pi}_X^* \oplus c_1(\mathcal{O}(1))\, \bar{\pi}_X^* : \mathscr{F}(i)(X) \oplus \mathscr{F}(i-1)(X)[-2] \longrightarrow \mathscr{F}(i)(\mathbb{P}^1_X)$$

is an equivalence of spectra.

Remark 3.11. In all the situations of interest, the map c_1 in Definition 3.10 is induced by a suitable notion of first Chern class, corresponding to an orientation for the family of presheaves $\{\mathscr{F}(i)\}_{i\in\mathbb{Z}}$ ([AI23, Section 3]).

Lemma 3.12 ([CTHK97])**.** *Let S be a qcqs scheme, and $\mathscr{F}(i)$, $i \geqslant 0$, be spectra-valued presheaves on smooth S-schemes (resp. on qcqs S-schemes). If the family of presheaves $\{\mathscr{F}(i)\}_{i\geqslant 0}$ satisfies the $\mathbb{P}^1$-bundle formula, then for every integer $i \geqslant 0$, the presheaf $\mathscr{F}(i)$ is deflatable.*

Proof. The proof is the same as in [CTHK97, Proposition 5.4.3] (see also [EM23, Lemma 6.12 and Remark 6.13]). More precisely, consider the diagram of spectra

$$\begin{array}{ccc} \mathscr{F}(i)(\mathbb{P}^1_X) & \xrightarrow{\quad j_X^* \quad} & F(i)(\mathbb{A}^1_X) \\ \uparrow\wr & \searrow^{\infty_X^*} & \uparrow{\scriptstyle \pi_X^*} \\ \mathscr{F}(i)(X) \oplus \mathscr{F}(i-1)(X)[-2] & & \mathscr{F}(i)(X), \end{array}$$

where the left map is an equivalence by the $\mathbb{P}^1$-bundle formula. We want to prove that the right triangle commutes. It suffices to do so after pre-composing with the left equivalence, and to check the commutativity on each of the factors. On the first factor, the commutativity follows by functoriality of $\mathscr{F}(i)$. On the second factor, it is a consequence of the functoriality of the map c_1, and of the identifications $j_X^*(\mathcal{O}_{\mathbb{P}^1_X}(1)) \cong \mathcal{O}_{\mathbb{A}^1_X}$ and $\infty_X^*(\mathcal{O}_{\mathbb{P}^1_X}(1)) \cong \mathcal{O}_X$. □

Corollary 3.13. *Let P be a Prüfer ring (e.g., a valuation ring), $\mathscr{F}\colon \mathrm{Sch}_P^{\mathrm{qcqs,op}} \to \mathrm{Sp}$ be a finitary Nisnevich sheaf, and $j \in \mathbb{Z}$ be an integer satisfying the following conditions:*

(i) the presheaf $\mathscr{F}$ is deflatable in the sense of Definition 3.6;

(ii) for any henselian valuation ring V which is an ind-smooth P-algebra, we have $\pi_j(\mathscr{F}(V)) = 0$.

Then, for any henselian local ind-smooth P-algebra R, we have $\pi_j(\mathscr{F}(R)) = 0$.

Proof. This is a consequence of Theorem 3.5 where we take F to be the Nisnevich sheafification of the presheaf $\pi_j(\mathscr{F}(-))$, which satisfies the Horrocks principle by Remark 3.8. □

Remark 3.14. If P is a field (resp. a discrete valuation ring), then the local ring R_η appearing in Theorem 3.2 is also a field (resp. a discrete valuation ring). As a consequence, the results of this section actually extend the results of Colliot-Thélène–Hoobler–Khan [CTHK97] (resp. of Gillet–Levine [GL87]).

Corollary 3.15. *Let P be a Prüfer ring, and $\{E(i)\}_{i\in\mathbb{Z}} \in \mathrm{MS}(P)$ be a $\mathbb{P}^1$-motivic spectrum in the sense of [AHI24]. Then for all integers $i, j \in \mathbb{Z}$, if the cohomology presheaf $\mathrm{H}^j(E(i)(-))$ vanishes on henselian valuation rings which are ind-smooth P-algebras, then it vanishes on all henselian local ind-smooth P-algebras.*

Proof. The presheaf $\mathrm{H}^j(E(i)(-))$ is deflatable by [AHI25, Corollaries 4.4 and 4.11] (for oriented $\mathbb{P}^1$-motivic spectra, this is more directly a consequence of Lemma 3.12), so the result is a consequence of Corollary 3.13. □

4 Review of non-$\mathbb{A}^1$-invariant motivic cohomology

Given a qcqs scheme X, one may consider the associated spectra given by non-connective algebraic K-theory $\mathrm{K}(X)$, topological cyclic homology $\mathrm{TC}(X)$, Weibel's homotopy K-theory $\mathrm{KH}(X) := (L_{\mathbb{A}^1}\mathrm{K})(X)$, and cdh-local topological cyclic homology $(L_{\mathrm{cdh}}\mathrm{TC})(X)$. By a theorem of Kerz–Strunk–Tamme [KST18] and Land–Tamme [LT19], these invariants are related by a functorial cartesian square

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{TC}(X) \\ \downarrow & & \downarrow \\ \mathrm{KH}(X) & \longrightarrow & (L_{\mathrm{cdh}}\mathrm{TC})(X), \end{array}$$

where the top horizontal map is the cyclotomic trace map, and where the vertical maps are the canonical maps. One may then consider the following *motivic filtrations* on these invariants.

(1) The $\mathbb{N}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{cdh}}\mathrm{KH}(X)$, constructed by Bachmann–Elmanto–Morrow [BEM25] in terms of the classical $\mathbb{A}^1$-invariant motivic filtration on the algebraic K-theory of smooth affine $\mathbb{Z}$-schemes. We denote by $\mathbb{Z}(i)^{\mathrm{cdh}}(X) := \mathrm{gr}^i_{\mathrm{cdh}}\mathrm{KH}(X)[-2i]$ the shifted graded pieces of this filtration, which provide a theory of cdh-local motivic cohomology for X, where the cdh-topology is defined as in [EHIK21].

(2) The $\mathbb{Z}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC}(X)$, constructed by Elmanto–Morrow [EM23] and the first-named author [Bou24] in terms of the Hochschild–Kostant–Rosenberg filtration on negative cyclic homology [Ant19] and of the Bhatt–Morrow–Scholze filtration on p-completed topological cyclic homology [BMS19] (or, alternatively, in terms of the even filtration [HRW22]). We denote by $\mathbb{Z}(i)^{\mathrm{TC}}(X) := \mathrm{gr}^i_{\mathrm{mot}}\mathrm{TC}(X)[-2i]$ the shifted graded pieces of this filtration, which recover the Hodge-completed derived de Rham cohomology of X in characteristic zero, and the syntomic cohomology of X in positive characteristic.

(3) The $\mathbb{Z}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}L_{\mathrm{cdh}}\mathrm{TC}(X)$, introduced by Elmanto–Morrow [EM23] and the first-named author [Bou24] as the cdh sheafification $(L_{\mathrm{cdh}}\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{TC})(X)$ of the previous filtration. Henselian valuation rings being the local rings for the cdh topology, the structural properties of the shifted graded pieces $L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}(X)$ are usually determined by the value of $\mathbb{Z}(i)^{\mathrm{TC}}$ at henselian valuation rings.

These three filtrations are multiplicative (*i.e.*, have a structure of filtered $\mathbb{E}_\infty$-rings), and are functorial in X. Given these filtrations, Elmanto–Morrow [EM23] and the first-named author [Bou24] then proved that there exists a multiplicative, functorial, $\mathbb{N}$-indexed filtration $\mathrm{Fil}^{\star}_{\mathrm{mot}}\mathrm{K}(X)$ on $\mathrm{K}(X)$ such that the previous cartesian square upgrades to a filtered cartesian square. We denote by $\mathbb{Z}(i)^{\mathrm{mot}}(X) := \mathrm{gr}^i_{\mathrm{mot}}\mathrm{K}(X)[-2i]$ the shifted graded pieces of this filtration, which provide a theory of motivic cohomology for X. By construction, the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ are related to the previous invariants by a functorial cartesian square

$$\begin{array}{ccc} \mathbb{Z}(i)^{\mathrm{mot}}(X) & \longrightarrow & \mathbb{Z}(i)^{\mathrm{TC}}(X) \\ \downarrow & & \downarrow \\ \mathbb{Z}(i)^{\mathrm{cdh}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\,\mathbb{Z}(i)^{\mathrm{TC}}\big)(X) \end{array}$$

in the derived category $\mathcal{D}(\mathbb{Z})$. In the rest of this section, we review the results about the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ that will be used in later sections.

Theorem 4.1 ([EM23, Bou24])**.** *The functors*

$$\mathbb{Z}(i)^{\mathrm{mot}} \colon \mathrm{Sch}^{\mathrm{qcqs,op}} \longrightarrow \mathcal{D}(\mathbb{Z}), \quad i \geqslant 0$$

are finitary Nisnevich sheaves ([Bou24, Corollary 4.60]), and satisfy the following properties for every qcqs scheme X.

(1) The $\mathbb{P}^1$-bundle formula, i.e., the natural maps

$$\pi^* \oplus c_1^{\mathrm{mot}}(\mathcal{O}(1))\pi^* \colon \mathbb{Z}(i)^{\mathrm{mot}}(X) \oplus \mathbb{Z}(i-1)^{\mathrm{mot}}(X)[-2] \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{P}^1_X), \quad i \geqslant 0,$$

where $\pi \colon \mathbb{P}^1_X \to X$ is the canonical projection map, are equivalences in the derived category $\mathcal{D}(\mathbb{Z})$ ([Bou24, Theorem 8.7]). More generally, the functors $\mathbb{Z}(i)^{\mathrm{mot}}$ satisfy the projective bundle formula ([Bou24, Theorem 8.16]).

(2) There exists a natural equivalence

$$\mathrm{K}(X)_{\mathbb{Q}} \simeq \bigoplus_{i \geqslant 0} \mathbb{Q}(i)^{\mathrm{mot}}(X)[2i]$$

in the derived category $\mathcal{D}(\mathbb{Q})$, induced by suitable Adams operations on the rational algebraic K-theory $\mathrm{K}(X)_{\mathbb{Q}}$ ([Bou24, Corollary 4.61]).

(3) For every prime number p and every integer $k \geqslant 1$, there is a natural fibre sequence

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) \longrightarrow \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([Bou24, Theorem 5.10]). In particular, the left map of this fibre sequence is an isomorphism in degrees less than or equal to i [Bou24, Corollary 5.11].

In Theorem 4.1 (3), we use the *syntomic realisation map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

from the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) := \mathbb{Z}(i)^{\mathrm{mot}}(X) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^k$ to Bhatt–Lurie's syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$ [BL22, Section 8.4], as constructed in [Bou24, Construction 5.8]. Taking fibres along the horizontal maps in the natural commutative diagram

$$\begin{array}{ccc}
\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) & \longrightarrow & \big(L_{\mathrm{Nis}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle L_{\mathrm{cdh}}} \\
\mathbb{Z}/p^k(i)^{\mathrm{syn}}(X) & \longrightarrow & \big(L_{\mathrm{cdh}}\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X),
\end{array}$$

Theorem 4.1 (3) then induces, for any qcqs scheme X, a natural map

$$\big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \tag{4.1.1}$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$, which we call the *Beilinson–Lichtenbaum comparison map*. When X is smooth over a Dedekind domain, using the Beilinson–Lichtenbaum conjecture for Bloch's cycle complexes ([Gei04]), this comparison map is the reduction mod p^k of the comparison map from Bloch's cycle complexes to the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ ([Bou24, Definition 3.22]). More generally, we will use this Beilinson–Lichtenbaum comparison map to formulate our Theorem 5.1. Also note that, for any qcqs scheme X, the composite of the Beilinson–Lichtenbaum comparison map with the syntomic realisation map

$$\big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

is an isomorphism in degrees less than or equal to i, since both maps of this composite are actually isomorphisms in this range by Theorem 4.1 (3).

A key point in our arguments will be to use results in classical motivic cohomology to understand the behaviour of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ at valuation rings.

Remark 4.2 (Motivic cohomology of valuation rings, [Bou24, Section 11.3])**.** For every henselian valuation ring V and every integer $i \geqslant 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(V) \in \mathcal{D}(\mathbb{Z})$ lives in degrees at most i.

In order to compare the Milnor range of integral motivic cohomology with its mod p^k counterpart, we will need the following vanishing result.

Remark 4.3 (Motivic cohomology of henselian local rings, [Bou24, Section 7])**.** For every henselian local ring R and every integer $i \geqslant 0$, the motivic cohomology group $\mathrm{H}^j_{\mathrm{mot}}(R, \mathbb{Z}(i)) := \mathrm{H}^j(\mathbb{Z}(i)^{\mathrm{mot}}(R))$ is zero for $j = i+1$ ([Bou24, Corollary 7.10]), and torsion for $i < j \leqslant 2i$ ([Bou24, Proposition 7.4]).

Remark 4.4 (Étale motivic cohomology)**.** Although algebraic K-theory of schemes does *not* satisfy étale descent (see for instance [CM21]), one may consider two of its étale-local variants. The first one is étale K-theory $\mathrm{K}^{\mathrm{\acute{e}t}}$, defined here as the étale sheafification of algebraic K-theory. Alternatively, one may also consider Selmer K-theory $\mathrm{K}^{\mathrm{Sel}}$, introduced by Clausen [Cla17] and Clausen–Mathew [CM21] as the pullback $L_{\mathrm{KU}}\mathrm{K} \times_{L_{\mathrm{KU}}\mathrm{TC}} \mathrm{TC}$, where L_{KU} denotes the Bousfield localisation at the complex K-theory spectrum KU. Selmer K-theory, has the advantage of being defined at the level of categories, and is even a localising invariant in the sense of [BGT13]. Moreover, it admits a natural map $\mathrm{K}^{\mathrm{\acute{e}t}} \to \mathrm{K}^{\mathrm{Sel}}$ which is an isomorphism on homotopy in degrees $\geqslant -1$ ([CM21, Theorem 1.1]), and is the étale localisation of algebraic K-theory when seen in the category of $\mathbb{P}^1$-motivic spectra ([AI23, Theorems 0.1.1 and 5.4.4]).

Similarly, at the level of motivic cohomology, the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ are represented by a $\mathbb{P}^1$-motivic spectrum $\mathrm{H}\mathbb{Z}^{\mathrm{mot}}$ ([Bou24, Corollary 8.17]), and one may consider either the étale motivic complexes $\mathbb{Z}(i)^{\mathrm{\acute{e}t}}$, defined as the étale sheafification of the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$, or the Selmer motivic complexes $\mathbb{Z}(i)^{\mathrm{Sel}}$, defined as the cohomology represented by the étale localisation $\mathrm{H}\mathbb{Z}^{\mathrm{Sel}}$ in $\mathbb{P}^1$-motivic spectra of $\mathrm{H}\mathbb{Z}^{\mathrm{mot}}$. By construction, these two cohomology theories correspond to the shifted graded pieces of corresponding filtrations on étale K-theory and on Selmer K-theory, respectively. The Selmer motivic complexes $\mathbb{Z}(i)^{\mathrm{Sel}}$ admit a natural comparison map $\mathbb{Z}(i)^{\mathrm{\acute{e}t}} \to \mathbb{Z}(i)^{\mathrm{Sel}}$, and are naturally identified, with mod p^k coefficients, with Bhatt–Lurie's syntomic complexes $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ ([BL22, Remark 8.4.3] and [AHI24, Example 9.21]).

5 Proof of the Beilinson–Lichtenbaum phenomenon

In this section, we prove Theorems B and C, using Theorem 3.5. Let us fix a prime number p throughout the section.

Theorem 5.1 (Beilinson–Lichtenbaum over Prüfer rings)**.** *Let X be an ind-smooth scheme over a Prüfer ring P. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, the Beilinson–Lichtenbaum comparison map* (4.1.1)

$$\big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\big)(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. The presheaves $L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ and $\mathbb{Z}/p^k(i)^{\mathrm{mot}}$ are finitary Nisnevich sheaves, so it suffices to prove the result on henselian local ind-smooth P-algebras. For such a ring R, the statement amounts to proving that the natural map

$$\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(R)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. By Theorem 4.1 (3), this map is an equivalence in degrees less than or equal to i, so it suffices to prove that for every integer $j > i$, the abelian group $\mathrm{H}^j(\mathbb{Z}/p^k(i)^{\mathrm{mot}}(R))$ is zero.

Let $j \in \mathbb{Z}$ be an integer such that $j > i$. By Remark 4.2, this vanishing holds if R is a henselian valuation ring. Moreover, the presheaves $\mathbb{Z}/p^k(i)^{\mathrm{mot}}$ are finitary Nisnevich sheaves that satisfy the $\mathbb{P}^1$-bundle formula (Theorem 4.1 (1)), and thus, are deflatable (Lemma 3.12). The vanishing in general is then a consequence of Gersten's injectivity (Theorem 3.5). ☐

Remark 5.2. The statement of Theorem 5.1 is not expected to hold over an arbitrary base ring. Indeed, such a statement would imply, via the Atiyah–Hirzebruch spectral sequence, that non-connective algebraic K-theory (with finite coefficients) is the Nisnevich sheafification of connective algebraic K-theory on arbitrary schemes, which is known to be true only after also enforcing the projective bundle formula [AI23]. However, we expect that the statement of Theorem 5.1 should also hold over an arbitrary regular base ring.

Corollary 5.3. *Let X be an ind-smooth scheme over a Prüfer ring. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X)$ lives, Nisnevich-locally on X, in degrees at most i.*

Proof. This is a consequence of Theorem 5.1. ☐

Corollary 5.4. *For every Prüfer ring P and every integer $k \geqslant 1$, the family of presheaves*

$$\{L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}\}_{i\geqslant 0}$$

satisfies the projective bundle formula on ind-smooth P-schemes.

Proof. The family of presheaves $\{\mathbb{Z}/p^k(i)^{\mathrm{mot}}\}_{i\geqslant 0}$ satisfies the projective bundle formula on all qcqs schemes (Theorem 4.1 (1)), so this is a consequence of Theorem 5.1. ☐

Corollary 5.5. *Let $f : P \to P'$ be a morphism of Prüfer rings, and $\mathrm{H}\mathbb{Z}^{\mathrm{mot}}_P \in \mathrm{MS}(P)$ and $\mathrm{H}\mathbb{Z}^{\mathrm{mot}}_{P'} \in \mathrm{MS}(P')$ be the $\mathbb{P}^1$-motivic spectra representing the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}$ over P and P' respectively. Then the natural map*

$$f^*\mathrm{H}\mathbb{Z}^{\mathrm{mot}}_P \longrightarrow \mathrm{H}\mathbb{Z}^{\mathrm{mot}}_{P'}$$

is an equivalence in $\mathrm{MS}(P')$.

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. The result modulo p is a consequence of Theorem 5.1 and Corollary 5.4, where we use the fact that the functor $\tau^{\leqslant i}\,\mathbb{F}_p(i)^{\mathrm{syn}} : \mathrm{Rings} \to \mathcal{D}(\mathbb{Z}/p^k)$ is left Kan extended from smooth $\mathbb{Z}$-algebras ([Bou24, Proposition 7.7]). The result rationally holds true for any morphism of commutative rings $f : P \to P'$, as a consequence of the Adams decomposition (Theorem 4.1 (2)) and of base change for algebraic K-theory in $\mathbb{P}^1$-motivic spectra ([AI23, Theorem 0.1.1]). ☐

In the next results, we consider the étale motivic complexes $\mathbb{Z}(i)^{\text{ét}}$ and $\mathbb{Z}(i)^{\mathrm{Sel}}$ introduced in Remark 4.4.

Corollary 5.6. *Let X be an ind-smooth scheme over a Prüfer ring. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, the natural map*

$$\mathbb{Z}/p^k(i)^{\text{ét}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. By [AMMN22, Theorem 5.1 (1) and Corollary 5.43], the weight-i Bhatt–Morrow–Scholze's syntomic complex lives, étale-locally on any qcqs scheme X, in degrees at most i. By [BL22, Remark 8.4.4], the syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ then satisfies the same property, hence the desired result is a consequence of Theorem 5.1. ☐

Proposition 5.7. *Let X be an ind-smooth scheme over a Prüfer ring. Then for every integer $i \geqslant 0$, the natural map*

$$\mathbb{Z}(i)^{\text{ét}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{Sel}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. It suffices to prove the result rationally, and modulo p for every prime number p. The result modulo p is Corollary 5.6. The result rationally holds true for any qcqs scheme X, as a consequence of the Adams decomposition (Theorem 4.1 (2)) and of the fact that the natural map $\mathrm{K}(X) \to \mathrm{K}^{\mathrm{Sel}}(X)$ is a rational equivalence ([CM21, Example 6.2]). ☐

Theorem 5.8. *Let X be an ind-smooth scheme over a Prüfer ring. Then for every integer $i \geqslant 0$, the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ lives, Nisnevich-locally on X, in degrees at most i, and the induced natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}(i)^{\text{ét}}\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. Let R be the henselisation of a local ring of X. The ring R is then a henselian local ind-smooth algebra over a valuation ring, so the motivic complex $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(R)$ lives in degrees at most i (Corollary 5.3). By the short exact sequence of abelian groups

$$0 \longrightarrow \mathrm{H}^j_{\mathrm{mot}}(R, \mathbb{Z}(i))/p^k \longrightarrow \mathrm{H}^j_{\mathrm{mot}}(R, \mathbb{Z}/p^k(i)) \longrightarrow \mathrm{H}^{j+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))[p^k] \longrightarrow 0,$$

for every prime number p and every integer $j \geqslant 0$, this implies that the abelian group $\mathrm{H}^j_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is torsionfree for $j \geqslant i+2$. By Remark 4.3, where we use that R is henselian local, the abelian group $\mathrm{H}^{i+1}_{\mathrm{mot}}(R, \mathbb{Z}(i))$ is zero, hence torsionfree. It then suffices to prove that the rational motivic complex $\mathbb{Q}(i)^{\mathrm{mot}}(R)$ lives in degrees at most i. This complex vanishes in degrees $i+1$ to $2i$ by Remark 4.3, and is a direct summand of the shift $\mathrm{K}(R)_{\mathbb{Q}}[-2i]$ of the rational algebraic K-theory of R (Theorem 4.1 (2)). Moreover, by [AMM22, Corollary 2.3 and Proposition 2.4.1], the negative K-groups of ind-smooth algebras over valuation rings are zero. In particular, the negative K-groups of R are zero, and the cohomology groups of the complex $\mathbb{Q}(i)^{\mathrm{mot}}(R)$ also vanish in degrees more than $2i$. This proves the first claim. In particular, the canonical map $\mathbb{Z}(i)^{\mathrm{mot}}(X) \to \mathbb{Z}(i)^{\mathrm{\acute{e}t}}(X)$ naturally factors as

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Z}(i)^{\mathrm{\acute{e}t}}\big)(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{\acute{e}t}}(X)$$

in the derived category $\mathcal{D}(\mathbb{Z})$.

We prove now that the first map of this composite is an equivalence. It suffices to do so rationally and modulo p for every prime number p. By Thomason ([Tho85, Proposition 2.14]), rational algebraic K-theory satisfies étale descent, therefore, the same holds for rational motivic cohomology (Theorem 4.1 (2)). The claim that the natural map $\mathbb{Q}(i)^{\mathrm{mot}}(X) \to \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{Q}(i)^{\mathrm{mot}}\big)(X)$ is an equivalence is then a consequence of the previous paragraph. Modulo p, and using again that the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$ vanishes in degree $i+1$ for henselian local rings (Remark 4.3), the desired statement is equivalent to the fact that the natural map

$$\mathbb{F}_p(i)^{\mathrm{mot}}(X) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\,\mathbb{F}_p(i)^{\mathrm{\acute{e}t}}\big)(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$. After using Corollary 5.6 to express the $\mathbb{F}_p(i)^{\mathrm{\acute{e}t}}$ in terms of syntomic cohomology, this is a consequence of Theorem 5.1. ☐

Proof of Theorem C. The henselian local ring R is local for the Nisnevich topology, and is ind-smooth over a valuation ring V by definition, so the Beilinson–Lichtenbaum comparison map induces a natural equivalence

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(R) \simeq \tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)$$

in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ (Theorem 5.1). If the commutative ring V is F-smooth, then ind-smooth algebras over V are F-smooth ([BM23, Propositions 4.6 and 4.9]). The desired identification of the complex $\tau^{\leqslant i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R)$ in terms of étale cohomology, for p-torsionfree F-smooth rings R, is then [BM23, Theorem 1.8], whose proof relies on absolute prismatic cohomology. ☐

We end this section with two consequences of Theorem 5.1, which will be used in Remark 6.3.

Corollary 5.9. *Let V be a p-adically separated valuation ring of residue characteristic p whose p-completion is perfectoid (e.g., a perfectoid valuation ring of residue characteristic p), F be the fraction field of V, and X be an ind-smooth scheme over V. Then for all integers $i \geqslant 0$ and $k \geqslant 1$, the natural map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X_F)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$.

Proof. If V is an $\mathbb{F}_p$-algebra, then V is a filtered colimit of smooth $\mathbb{F}_p$-algebras, and the result is a consequence of Geisser–Levine's description of the p-adic motivic cohomology of smooth $\mathbb{F}_p$-schemes (see [AMM22, Remark 3.5]). We assume now that p is nonzero in V. In particular, because V is p-adically separated, we know that $V[\frac{1}{p}] = F$ ([FK18, Chapter 0, Proposition 6.7.2]), and it then suffices to prove that the natural map

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{mot}}(X[\tfrac{1}{p}])$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$. Perfectoid rings are F-smooth ([BL22, Proposition 4.12]), so V is a p-torsionfree F-smooth valuation ring. By Theorem C, it then suffices to prove that the natural map $\mathrm{H}^i_{\mathrm{mot}}(X,\mathbb{Z}/p^k(i)) \to \mathrm{H}^i_{\mathrm{mot}}(X[\frac{1}{p}],\mathbb{Z}/p^k(i))$ is an isomorphism of abelian groups. The presheaves $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-)$ and $\mathbb{Z}/p^k(i)^{\mathrm{mot}}(-[\frac{1}{p}])$ are deflatable finitary Nisnevich sheaves on qcqs V-schemes, so it suffices to prove that the natural map

$$\mathrm{H}^i_{\mathrm{mot}}(V',\mathbb{Z}/p^k(i)) \longrightarrow \mathrm{H}^i_{\mathrm{mot}}(V'[\tfrac{1}{p}],\mathbb{Z}/p^k(i))$$

is an isomorphism for every henselian valuation ring V' which is ind-smooth over V (Theorem 3.5). This map of abelian groups is injective by Theorem 4.1 (3) and [Bou23, Corollary 3.2] (whose proof holds for general valuation rings extensions V' of a perfectoid valuation ring V, but is considerably simplified when V' is ind-smooth over V). The symbol maps in motivic cohomology ([Bou24, Section 7.2]) naturally fit in a commutative diagram

$$\begin{array}{ccc} (V'^{\times}/p^k)^{\otimes i} & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(V',\mathbb{Z}/p^k(i)) \\ \downarrow & & \downarrow \\ (V'[\frac{1}{p}]^{\times}/p^k)^{\otimes i} & \longrightarrow & \mathrm{H}^i_{\mathrm{mot}}(V'[\frac{1}{p}],\mathbb{Z}/p^k(i)) \end{array}$$

of abelian groups. The bottom horizontal map is surjective by [BK86, Theorem 14.1]. The value group $\Gamma_{V'} := \mathrm{coker}(V'^{\times} \to V'[\frac{1}{p}]^{\times})$ of the valuation ring V' is naturally identified with that of the valuation ring V (see [Kun23, Lemma 3.10]). The value group of the p-adic completion of V is naturally identified with that of its tilt $V^\flat$, which is p-divisible, hence so is the value group of V ([FK18, Chapter 0, Theorem 9.1.1]). This implies that the left vertical map is an isomorphism for $i=1$, and hence for all $i \geqslant 0$. In particular, the right vertical map is surjective, which concludes the proof. ☐

Remark 5.10. Using Levine's absolute purity for motivic cohomology over discrete valuation rings, Corollary 5.9 was also recently proved independently by Annala and Elmanto in the special cases where V is either $\mathbb{Z}_p[p^{1/p^\infty}]$ or $\mathbb{Z}[\zeta_{p^\infty}]$ [AE25, Theorem 3.1 and Variant 3.4].

For smooth schemes over an algebraically closed field in which p is invertible, the following result was first proved by Suslin. We denote by $p\text{-}\mathrm{dim}(R)$ the p-dimension of a $\mathbb{F}_p$-algebra R, and refer to [KST21, Section 2] for the associated results needed in the following proof. In particular, note that if κ is a perfect $\mathbb{F}_p$-algebra, then $p\text{-}\mathrm{dim}(\kappa) = 0$.

Corollary 5.11. *Let $X \to S$ be an affine smooth morphism of schemes of relative dimension d. Then for every integer $k \geqslant 1$, the syntomic realisation map*

$$\mathbb{Z}/p^k(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}/p^k(i)^{\mathrm{syn}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ in each of the following cases:

(1) S is the spectrum of a separably closed field K of characteristic different from p, and $i \geqslant d$;

(2) S is the spectrum of a field κ of characteristic p, and $i \geqslant d + p\text{-}\mathrm{dim}(\kappa) + 1$;

(3) S is the spectrum of a rank one valuation ring V of mixed characteristic $(0,p)$ with separably closed fraction field K and residue field κ, and $i \geqslant d + p\text{-}\mathrm{dim}(\kappa) + 1$.

Proof. By Theorem 5.1, it suffices to prove in each case that the syntomic complex $\mathbb{Z}/p^k(i)^{\mathrm{syn}}$ lives, Nisnevich-locally on X, in degrees at most i.

(1) If p is invertible in the separably closed field K, then this is a consequence of the Artin–Grothendieck vanishing theorem [SGA4, exposé XIV, corollaire 3.2], which states in particular that the étale cohomology $R\Gamma_{\mathrm{\acute{e}t}}(R,\mu_{p^k}^{\otimes i})$ of any finite type K-algebra R of dimension d lives in degrees at most d, hence, in particular, in degrees at most i.

(2) If κ is of characteristic p, then $\tau^{>i}\,\mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) \simeq \widetilde{\nu}_k(i)(R)[-i-1]$ for every κ-algebra R ([AMMN22, Theorem 5.1 and Corollary 5.43]). If R is moreover of finite type over κ, then the p-dimension of R is equal

to $d + p\text{-dim}(\kappa)$. This implies that $\Omega^i_{R/\kappa}$ vanishes for every integer $i \geqslant d + p\text{-dim}(\kappa) + 1$, hence so does $\widetilde{\nu}_k(i)(R)$.

(3) Let R be the henselisation of a local ring of X. If p is invertible in R, then R is the henselisation of a local ring of $X[\frac{1}{p}]$, which is a smooth scheme of dimension d over the fraction field K of V, hence the claim reduces to (1). If p is not invertible in the henselian local ring R, then in particular the ring R is p-henselian, and its syntomic cohomology is naturally identified with Bhatt–Morrow–Scholze's syntomic cohomology ([BL22, Remark 8.4.4]). In particular, the natural map

$$\tau^{>i}\, \mathbb{Z}/p^k(i)^{\mathrm{syn}}(R) \longrightarrow \tau^{>i}\, \mathbb{Z}/p^k(i)^{\mathrm{syn}}(R \otimes_V \kappa)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z}/p^k)$ ([AMMN22, Theorem 5.2], see [Bou24, Theorem 2.27] for the precise version that we use). The ring $R/\mathfrak{m}R$ is the henselisation of a local ring of X_κ, which is a scheme of relative dimension d over the field κ, hence the claim again reduces to (2). □

6 $\mathbb{A}^1$-invariance and comparison with Bloch's cycle complexes

In this section, we prove that the motivic complexes $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ recover the classical motivic complexes

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) := \big(L_{\mathrm{Nis}}\, z^i(-,\bullet)\big)(X)[-2i]$$

on smooth schemes X over a Dedekind domain (Theorem 6.1). To formulate this result, we use the *classical-motivic comparison map* $\mathbb{Z}(i)^{\mathrm{cla}}(X) \to \mathbb{Z}(i)^{\mathrm{mot}}(X)$ of [Bou24, Definition 3.22]. We also denote by $\mathbb{Z}(i)^{\mathrm{cla}}$ the unique functorial extension of the previous Bloch cycle complex $\mathbb{Z}(i)^{\mathrm{cla}}$ to schemes that are ind-smooth over a Dedekind domain.

Theorem 6.1 (Comparison to classical motivic cohomology)**.** *Let X be an ind-smooth scheme over a Dedekind domain B. Then for every integer $i \geqslant 0$, the classical-motivic comparison map*

$$\mathbb{Z}(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. If the Dedekind domain B contains a field, then it is a filtered colimit of smooth algebras over this field, and the result is a consequence of [EM23, Corollary 6.4]. We assume now that the Dedekind domain B is of mixed characteristic.

It suffices to prove the result rationally, and modulo p for every prime number p. The result rationally is [Bou24, Proposition 6.2]. Let p be a prime number. By [Gei04, Corollary 4.4], Nisnevich-locally, the classical motivic complex $\mathbb{F}_p(i)^{\mathrm{cla}}$ lives in degrees at most i, so the natural composite map

$$\mathbb{F}_p(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathrm{syn}}(X)$$

naturally factors through a composite map

$$\mathbb{F}_p(i)^{\mathrm{cla}}(X) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\, \mathbb{F}_p(i)^{\mathrm{mot}}\big)(X) \longrightarrow \big(L_{\mathrm{Nis}}\tau^{\leqslant i}\, \mathbb{F}_p(i)^{\mathrm{syn}}\big)(X)$$

in the derived category $\mathcal{D}(\mathbb{F}_p)$. By [Gei04, Theorems 1.2 (2) and 1.3] and [BM23, Theorem 5.8 and Example 5.9], this composite map is an equivalence. Moreover, by Corollary 5.3, the middle term is naturally identified with the motivic complex $\mathbb{F}_p(i)^{\mathrm{mot}}(X)$. The second map is then an equivalence by Theorem 5.1, so the natural map

$$\mathbb{F}_p(i)^{\mathrm{cla}}(X) \longrightarrow \mathbb{F}_p(i)^{\mathrm{mot}}(X)$$

is also an equivalence in the derived category $\mathcal{D}(\mathbb{F}_p)$, as desired. □

Remark 6.2. When B is a field, Theorem 6.1 is [EM23, Corollary 6.4]. In mixed characteristic, Theorem 6.1 was also proved in the case of schemes of relative dimension one over a Dedekind domain in [Bou24, Section 6.2].

Remark 6.3 (Fontaine's crystalline conjecture)**.** Fontaine's crystalline conjecture [Fon82, conjecture A.11], first proved in general by Faltings [Fal89], asserts that for X a smooth proper scheme over a complete discrete valuation ring $\mathcal{O}_K$ of mixed characteristic $(0,p)$ and with perfect residue field k, there are natural isomorphisms of abelian groups

$$\mathrm{H}^n_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\text{crys}} \cong \mathrm{H}^n_{\text{crys}}(X_k/W(k))\otimes_{W(k)} B_{\text{crys}} \qquad \text{for all } n\geqslant 0,$$

compatible with filtrations, Galois actions, and Frobenius operators. As observed by Nizioł [Niz06], this comparison should be reminiscent of a suitable theory of motivic cohomology for mixed characteristic schemes. In this remark, we explain how the results of the present paper can be used to realise this expectation.

By [Niz98, Section 4] (see also [FM87], or [Fal88, 2.4]), the key point is to construct, at least for $i\gg 0$, a natural Galois-equivariant equivalence

$$R\Gamma_{\text{ét}}(X_{\overline{K}},\mu_{p^k}^{\otimes i}) \xrightarrow{\sim} \mathbb{Z}/p^k(i)^{\text{syn}}(X_{\mathcal{O}_{\overline{K}}}),$$

compatible with Poincaré duality and cycle classes, for every integer $k\geqslant 1$.[6] This equivalence can now be seen as the inverse of the bottom map in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^k(i)^{\text{mot}}(X_{\overline{K}}) & \longleftarrow & \mathbb{Z}/p^k(i)^{\text{mot}}(X_{\mathcal{O}_{\overline{K}}}) \\ \downarrow & & \downarrow \\ R\Gamma_{\text{ét}}(X_{\overline{K}},\mu_{p^k}^{\otimes i}) & \longleftarrow & \mathbb{Z}/p^k(i)^{\text{syn}}(X_{\mathcal{O}_{\overline{K}}}) \end{array}$$

where the horizontal maps are induced by functoriality of the motivic and syntomic complexes, and the vertical maps are instances of the syntomic realisation map [Bou24, Construction 5.8]. More precisely, for every smooth scheme $X_{\mathcal{O}_{\overline{K}}}$ over the valuation ring $\mathcal{O}_{\overline{K}}$, the vertical maps of this diagram are equivalences for $i\geqslant d+1$ by Proposition 5.11, and the top horizontal map is an equivalence for all $i\geqslant 0$ by Corollary 5.9. The compatibility with cycle classes is then a consequence of Theorem 6.1 (where we use that $\mathcal{O}_{\overline{K}}$ is a filtered union of discrete valuation rings), and the compatibility with Poincaré duality is in turn a consequence of the compatibility with cycle classes (see [Niz98, proof of Lemma 4.2]).

For the rest of this section, we use Theorem 3.5 to study the $\mathbb{A}^1$-invariance of the motivic complexes $\mathbb{Z}(i)^{\text{mot}}$.

Definition 6.4. A commutative ring R is *cdh-locally F-smooth* if every valuation ring V over R is F-smooth[7] in the sense of [BM23, Definition 1.7].

Examples 6.5. (1) A field is cdh-locally F-smooth. Indeed, valuation rings of characteristic zero are vacuously F-smooth, and valuation rings of positive characteristic are Cartier smooth by results of Gabber–Ramero and Gabber (see [KM21, Section 2]), hence F-smooth ([BM23, Proposition 4.14]).

(2) A perfectoid valuation ring is cdh-locally F-smooth. Indeed, valuation rings over a perfectoid valuation ring are F-smooth ([Bou23]).

(3) Any algebra over a cdh-locally F-smooth ring is cdh-locally F-smooth.

Remark 6.6. Conjecturally, all valuation rings are F-smooth, *i.e.*, $\mathbb{Z}$ is cdh-locally F-smooth. This conjecture is a consequence of Zariski's local uniformisation conjecture, which states that every valuation ring is a filtered colimit of regular local rings. Indeed, regular rings are F-smooth, and the class of F-smooth rings is stable under filtered colimits ([BM23]).

[6] In [Niz98, Theorem 4.1], and using algebraic K-theory, Nizioł constructs such an equivalence for X smooth projective of pure dimension d over $\mathcal{O}_K$, and with the restrictions that $i\geqslant \frac{16}{3}d^3+8d^2+\frac{14}{3}d$ and $p\geqslant 2i+\frac{16}{3}d^3+8d^2+\frac{8}{3}d+2$. While the latter restriction is partly explained by the use of Fontaine–Messing's syntomic cohomology, which works well integrally only in weights i less than p, the former restriction comes from Thomason's comparison between algebraic K-theory and étale cohomology.

[7] F-smoothness implicitly depends on a prime number p. Here we say that a valuation ring V is F-smooth if it is F-smooth for every prime number p. Note that valuation rings V are local, so that F-smoothness is vacuously true for almost all prime numbers p, *i.e.*, for those that are invertible in V.

The motivation for Definition 6.4 is the following result of Bachmann–Elmanto–Morrow, where the cdh-local motivic complex $\mathbb{Z}(i)^{\mathrm{cdh}}$ is the cdh sheafification of the motivic complex $\mathbb{Z}(i)^{\mathrm{mot}}$.

Theorem 6.7 ([BEM25])**.** *On qcqs schemes over a cdh-locally F-smooth ring B, the cdh-local motivic complexes $\mathbb{Z}(i)^{\mathrm{cdh}}$, $i \geqslant 0$, are $\mathbb{A}^1$-invariant and satisfy the $\mathbb{P}^1$-bundle formula.*

Remark 6.8. At the level of K-theory, the analogue of Theorem 6.7 holds for arbitrary qcqs schemes. More precisely, by a result of Kerz–Strunk–Tamme [KST18], KH-theory is the cdh-sheafification of K-theory on qcqs schemes.

The following result is a motivic analogue of the results of Antieau–Mathew–Morrow [AMM22, Section 2] on the $\mathbb{A}^1$-invariance of algebraic K-theory.

Theorem 6.9. *Let B be either a Dedekind domain or a cdh-locally F-smooth Prüfer domain, and X be an ind-smooth scheme over B. Then for all integers $i \geqslant 0$ and $m \geqslant 1$, the natural map*

$$\mathbb{Z}(i)^{\mathrm{mot}}(X) \longrightarrow \mathbb{Z}(i)^{\mathrm{mot}}(\mathbb{A}^m_X)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$.

Proof. If B is a Dedekind domain, this is a consequence of Theorem 6.1 and of the $\mathbb{A}^1$-invariance of Bloch cycle complexes.

Assuming now that B is a cdh-locally F-smooth Prüfer domain. In this case, it suffices to prove that the canonical map $\mathbb{Z}(i)^{\mathrm{mot}} \to \mathbb{Z}(i)^{\mathrm{cdh}}$ is an equivalence on ind-smooth schemes over B (Theorem 6.7). Since the Nisnevich finitary sheaves $\mathbb{Z}(i)^{\mathrm{mot}}$ and $\mathbb{Z}(i)^{\mathrm{cdh}}$ are both deflatable (Theorems 4.1 (1) and 6.7), the same is true for the fibre

$$\mathscr{F}(-) := \mathrm{fib}\big(\mathbb{Z}(i)^{\mathrm{mot}}(-) \to \mathbb{Z}(i)^{\mathrm{cdh}}(-)\big).$$

Hence, by Corollary 3.13, it suffices to show that $\mathscr{F}(V) = 0$ for any henselian valuation ring V over B. To prove this, note that henselian valuation rings are the local rings for the cdh topology, and that the presheaf $\mathbb{Z}(i)^{\mathrm{cdh}}$ is the cdh sheafification of the presheaf $\mathbb{Z}(i)^{\mathrm{mot}}$. In particular, for every henselian valuation ring V, the natural map

$$\mathbb{Z}(i)^{\mathrm{mot}}(V) \longrightarrow \mathbb{Z}(i)^{\mathrm{cdh}}(V)$$

is an equivalence in the derived category $\mathcal{D}(\mathbb{Z})$, *i.e.*, $\mathscr{F}(V) = 0$. □

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
Email address: tess.bouis@ur.de
URL: https://tessbouis.com

St. George Campus, University of Toronto, Toronto ON M5S 2E4, Canada
Email address: arnab.kundu@utoronto.ca
URL: https://arnabkundu.com